Probability Distributions: Multinomial and Poisson

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Multinomial distribution

- Recall: the binomial distribution is the number of successes from multiple Bernoulli success/fail events
- The **multinomial** distribution is the number of different outcomes from multiple categorical events
  - It is a generalization of the binomial distribution to more than two possible outcomes
  - As with the binomial distribution, each categorical event is assumed to be independent
  - **Bernoulli : binomial :: categorical : multinomial**
- Examples:
  - The number of times each face of a die turned up after 50 rolls
  - The number of times each suit is drawn from a deck of cards after 10 draws

Multinomial distribution

- Notation: let $\vec{X}$ be a vector of length K, where X_k is a random variable that describes the number of times that the kth value was the outcome out of N categorical trials.
  - The possible values of each X_k are integers from 0 to N
  - All X_k values must sum to N: $\sum_{k=1}^{K} X_k = N$
- Example: if we roll a die 10 times, suppose it comes up with the following values:

$$\vec{X} = <1, 0, 3, 2, 1, 3>$$

$$X_1 = 1$$
$$X_2 = 0$$
$$X_3 = 3$$
$$X_4 = 2$$
$$X_5 = 1$$
$$X_6 = 3$$

- The multinomial distribution is a joint distribution over multiple random variables: $P(X_1, X_2, \ldots, X_K)$

Multinomial distribution

- Suppose we roll a die 3 times. There are 216 (6^3) possible outcomes:

$$
\begin{aligned}
P(111) &= P(1)P(1)P(1) &= 0.00463 \\
P(112) &= P(1)P(1)P(2) &= 0.00463 \\
P(113) &= P(1)P(1)P(3) &= 0.00463 \\
P(114) &= P(1)P(1)P(4) &= 0.00463 \\
P(115) &= P(1)P(1)P(5) &= 0.00463 \\
P(116) &= P(1)P(1)P(6) &= 0.00463 \\
\dots & \quad \dots & \dots \\
P(665) &= P(6)P(6)P(5) &= 0.00463 \\
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\end{aligned}
$$

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Multinomial distribution

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Multinomial distribution

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- Example 2: $\vec{X} = < 0, 0, 1, 1, 1, 0 >$

Multinomial distribution

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- Example 2: $\vec{X} = < 0, 0, 1, 1, 1, 0 >$
    - $P(\vec{X}) = P(345) + P(354) + P(435) + P(453) + P(534) + P(543) = 0.02778$

Multinomial distribution

- The probability mass function for the multinomial distribution is:

$$f(\vec{x}) = \underbrace{\frac{N!}{\prod_{k=1}^{K} x_k!}}_{\text{Generalization of binomial coefficient}} \prod_{k=1}^{K} \theta_k^{x_k}$$

- Like categorical distribution, multinomial has a K-length parameter vector $\vec{\theta}$ encoding the probability of each outcome.
- Like binomial, the multinomial distribution has a additional parameter N, which is the number of events.

Multinomial distribution: summary

- Categorical distribution is multinomial when $N = 1$.
- Sampling from a multinomial: same code repeated N times.
  - Remember that each categorical trial is independent.
  - Question: Does this mean the count values (i.e., each X_1, X_2, etc.) are independent?

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- Remember this analogy:
  - **Bernoulli : binomial :: categorical : multinomial**

Poisson distribution

- We showed that the Bernoulli/binomial/categorical/multinomial are all related to each other
- Lastly, we will show something a little different
- The **Poisson** distribution gives the probability that an event will occur a certain number of times within a time interval
- Examples:
    - The number of goals in a soccer match
    - The amount of mail received in a day
    - The number of times a river will flood in a decade

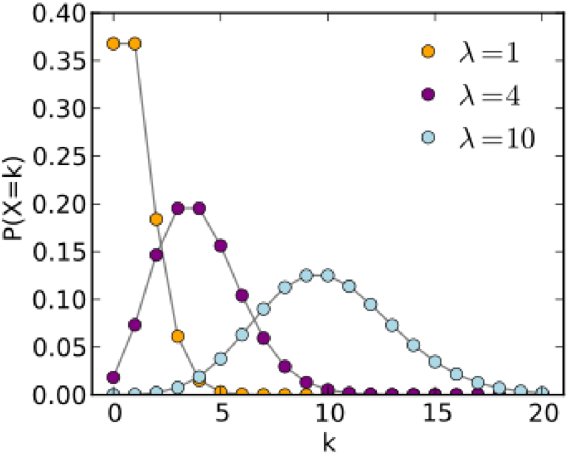
Poisson distribution

- Let the random variable X refer to the count of the number of events over whatever interval we are modeling.
  - X can be any positive integer or zero: $\{0, 1, 2, \ldots\}$
- The probability mass function for the Poisson distribution is:

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

- The Poisson distribution has one parameter λ, which is the average number of events in an interval.
  - $\mathbb{E}[X] = \lambda$

Poisson distribution



Poisson distribution

- Example: Poisson is good model of World Cup match having a certain number of goals
- A World Cup match has an average of 2.5 goals scored: $\lambda = 2.5$
-
  - $P(X = 0) = \frac{2.5^0 e^{-2.5}}{0!} = \frac{e^{-2.5}}{1} = 0.082$
  - $P(X = 1) = \frac{2.5^1 e^{-2.5}}{1!} = \frac{2.5e^{-2.5}}{1} = 0.205$
  - $P(X = 2) = \frac{2.5^2 e^{-2.5}}{2!} = \frac{6.25e^{-2.5}}{2} = 0.257$
  - $P(X = 3) = \frac{2.5^3 e^{-2.5}}{3!} = \frac{15.625e^{-2.5}}{6} = 0.213$
  - $P(X = 4) = \frac{2.5^4 e^{-2.5}}{4!} = \frac{39.0625e^{-2.5}}{24} = 0.133$

    $\dots$
  - $P(X = 10) = \frac{2.5^{10} e^{-2.5}}{10!} = \frac{9536.7432e^{-2.5}}{3628800} = 0.00022$

    $\dots$