# Probabilities and Data

Digging into Data: Jordan Boyd-Graber

University of Maryland

February 3, 2014

Slides adapted from Dave Blei and Lauren Hannah

# Roadmap

- What are probabilities
  - Discrete
  - Continuous
- How to manipulate probabilities
- Properties of probabilities

# Preface: Why make us do this?

- Probabilities are the language we use to describe data
- A reasonable (but geeky) definition of data science is how to get probabilities we care about from data
- Later classes will be about how to do this for different probability models and different types of data
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- But first, we need key definitions of probability
- So pay attention!
- Also, ya'll need to get your environments set up

# Outline

1. **Properties of Probability Distributions**
2. Working with probability distributions
3. Combining Probability Distributions
4. Continuous Distributions
5. Expectation and Entropy

# Random variable

- Probability is about *random variables.*
- A random variable is any “probabilistic” outcome.
- For example,
  - The flip of a coin
  - The height of someone chosen randomly from a population
- We’ll see that it’s sometimes useful to think of quantities that are not strictly probabilistic as random variables.
  - The temperature on 11/12/2013
  - The temperature on 03/04/1905
  - The number of times “streetlight” appears in a document

# Random variable

- Random variables take on values in a *sample space*.
- They can be *discrete* or *continuous*:
  - Coin flip: $\{H, T\}$
  - Height: positive real values $(0, \infty)$
  - Temperature: real values $(-\infty, \infty)$
  - Number of words in a document: Positive integers $\{1, 2, \dots\}$
- We call the outcomes *events*.
- Denote the random variable with a capital letter; denote a realization of the random variable with a lower case letter.
- E.g., $X$ is a coin flip, $x$ is the value ($H$ or $T$) of that coin flip.

# Discrete distribution

- A discrete distribution assigns a probability to every event in the sample space
- For example, if $X$ is an (unfair) coin, then

$$\begin{aligned} P(X=H) &= 0.7 \\ P(X=T) &= 0.3 \end{aligned}$$

- And probabilities have to be greater than 0
- Probabilities of disjunctions are sums over part of the space. E.g., the probability that a die is bigger than 3:

$$P(D>3)=P(D=4)+P(D=5)+P(D=6)$$

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An *event* is a set of outcomes to which a probability is assigned

- drawing a black card from a deck of cards
- drawing a King of Hearts

Intersections and unions:

- Intersection: drawing a red and a King

$$P(A \cap B) \tag{1}$$

- Union: drawing a spade or a King

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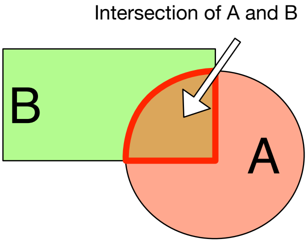

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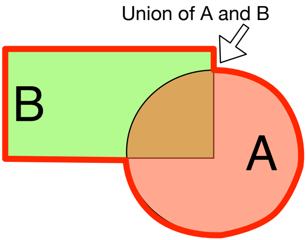

# Joint distribution

- Typically, we consider collections of random variables.
- The joint distribution is a distribution over the configuration of all the random variables in the ensemble.
- For example, imagine flipping 4 coins. The joint distribution is over the space of all possible outcomes of the four coins.

$$
\begin{aligned}
P(HHHH) &= 0.0625 \\
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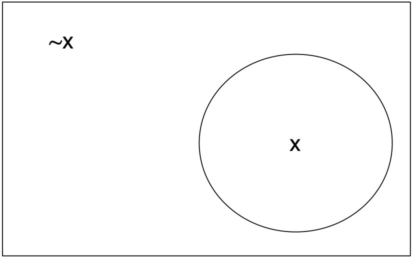
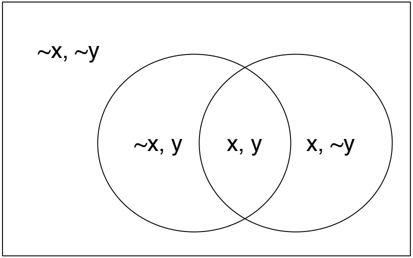
&\dots
\end{aligned}
$$

- You can think of it as a single random variable with 16 values.

# Visualizing a joint distribution



# Marginalization

If we are given a joint distribution, what if we are only interested in the distribution of one of the variables?

We can compute the distribution of $P(X)$ from $P(X, Y, Z)$ through *marginalization*:

$$\begin{aligned}\sum_y \sum_z P(X, Y = y, Z = z) &= \sum_y \sum_z P(X) P(Y = y, Z = z \,|\, X) \\ &= P(X) \sum_y \sum_z P(Y = y, Z = z \,|\, X) \\ &= P(X)\end{aligned}$$

# Marginalization (from Leyton-Brown)

**Joint distribution**

temperature (T) and weather (W)

| | T=Hot | T=Mild | T=Cold |
|---|---|---|---|
| W=Sunny | .10 | .20 | .10 |
| W=Cloudy | .05 | .35 | .20 |

Marginalization allows us to compute distributions over smaller sets of variables:

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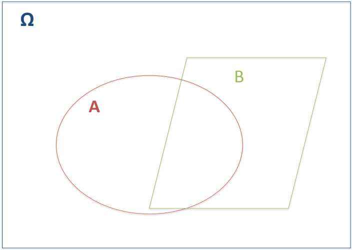

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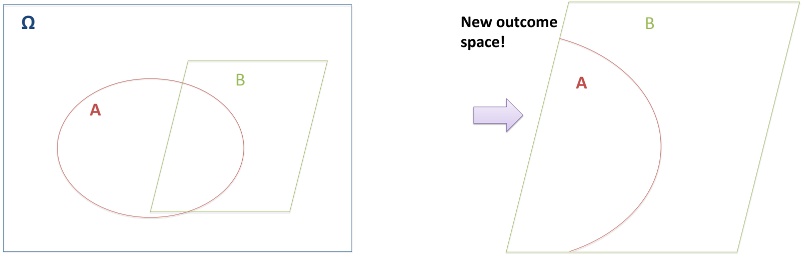

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$$= \frac{1}{9}$$

# Outline

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- For example, let $Y$ be a disease and $X$ be a symptom. We may know $P(X|Y)$ and $P(Y)$ from data. Use the chain rule to obtain the probability of having the disease and the symptom.
- In general, for any set of $N$ variables

$$P(X_1,\ldots,X_N) = \prod_{n=1}^{N} P(X_n|X_1,\ldots,X_{n-1})$$

# Bayes' Rule

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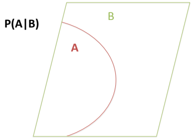

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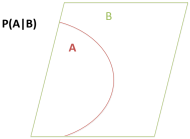


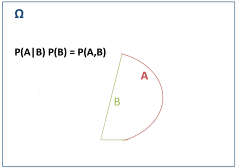

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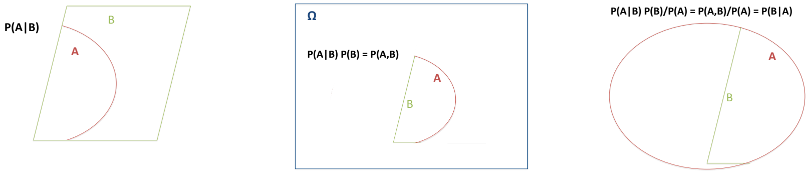

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Conditional probabilities equal unconditional probabilities with independence:

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- *Knowing $Y$ tells us nothing about $X$*

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Mathematical examples:

- If I draw two socks from my (multicolored) laundry, is the color of the first sock independent from the color of the second sock?

# Independence

Random variables $X$ and $Y$ are independent if and only if $P(X = x, Y = y) = P(X = x)P(Y = y)$.

Conditional probabilities equal unconditional probabilities with independence:

- $P(X = x \mid Y) = P(X = x)$
- *Knowing $Y$ tells us nothing about $X$*

Mathematical examples:

- If I draw two socks from my (multicolored) laundry, is the color of the first sock independent from the color of the second sock?
- If I flip a coin twice, is the first outcome independent from the second outcome?

# Independence

Intuitive Examples:

- Independent:
  - you use a Mac / the Green line is on schedule
  - snowfall in the Himalayas / your favorite color is blue

# Independence

Intuitive Examples:

- Independent:
  - you use a Mac / the Green line is on schedule
  - snowfall in the Himalayas / your favorite color is blue
- Not independent:
  - you vote for Mitt Romney / you are a Republican
  - there is a traffic jam on the Beltway / the Redskins are playing

# Independence

Sometimes we make convenient assumptions.

- the values of two dice
- the value of the first die and the sum of the values
- whether it is raining and the number of taxi cabs
- whether it is raining and the amount of time it takes me to hail a cab
- the first two words in a sentence

# Outline

# Continuous random variables

- We've only used discrete random variables so far (e.g., dice)
- Random variables can be continuous.
- We need a *density* $p(x)$, which *integrates* to one.
  E.g., if $x \in \mathbb{R}$ then
  $$\int_{-\infty}^{\infty} p(x)dx = 1$$
- Probabilities are integrals over smaller intervals. E.g.,
  $$P(X \in (-2.4, 6.5)) = \int_{-2.4}^{6.5} p(x)dx$$
- Notice when we use $P$, $p$, $X$, and $x$.

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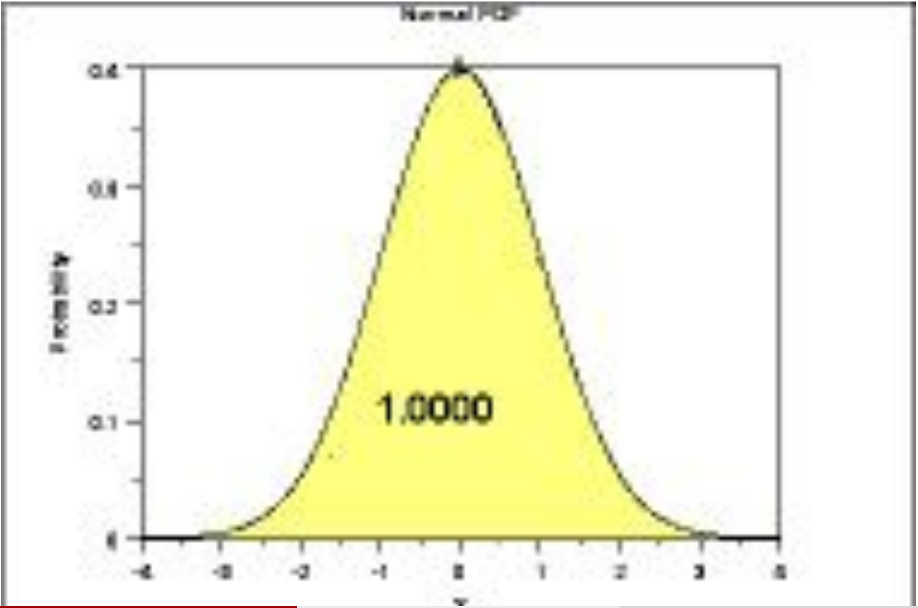
- Integrals? I didn't sign up for this!

# Integrals?



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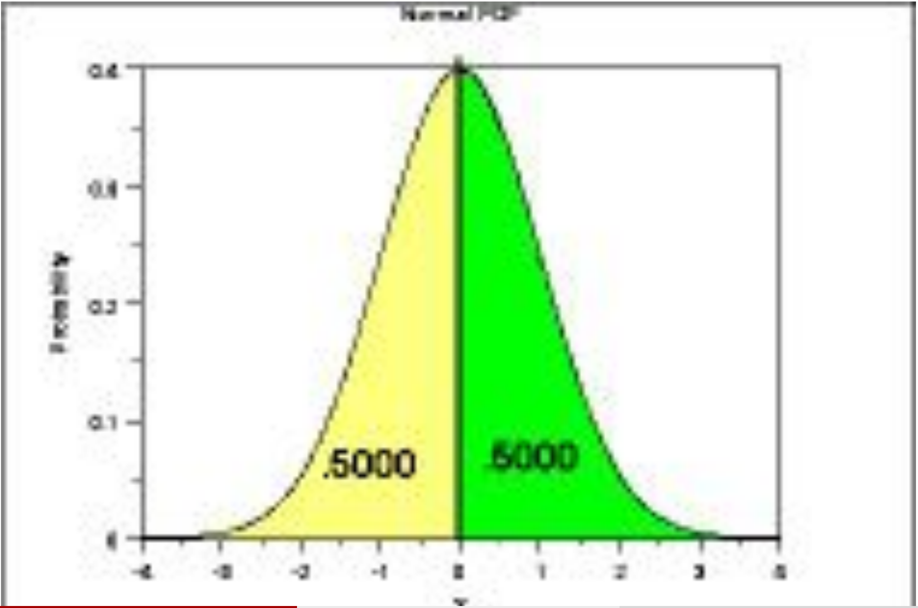

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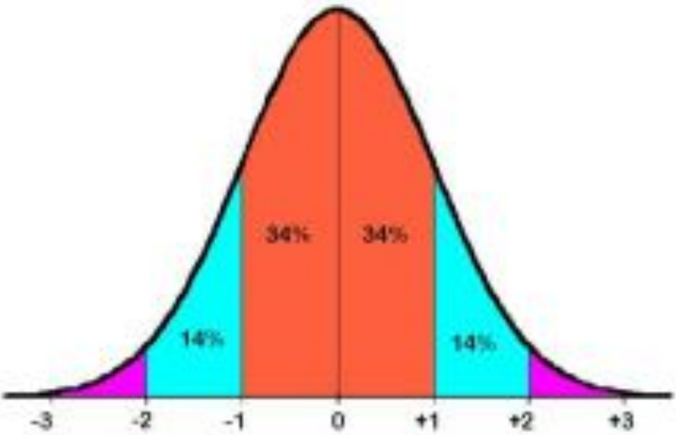


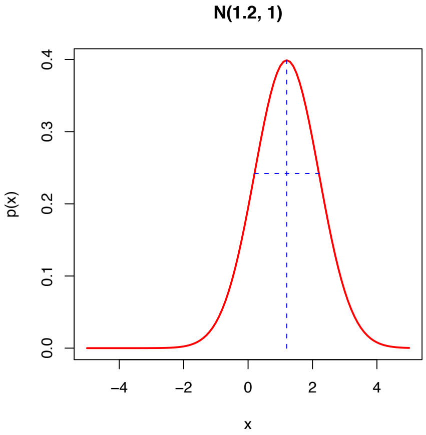
# The Gaussian distribution

- The Gaussian (or Normal) is a continuous distribution.

$$p(x|\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

- The density of a point $x$ is proportional to the negative exponentiated half distance to $\mu$ scaled by $\sigma^2$.
- $\mu$ is called the *mean*; $\sigma^2$ is called the *variance*.

# Gaussian density




- The mean $\mu$ controls the location of the bump.
- The variance $\sigma^2$ controls the spread of the bump.

# Outline

# Expectation

An *expectation* of a random variable is a weighted average:

$$\mathrm{E}[f(X)] = \sum_{x=1}^{\infty} f(x)\,p(x) \qquad \text{(discrete)}$$

$$= \int_{-\infty}^{\infty} f(x)\,p(x)\,dx \qquad \text{(continuous)}$$

# Expectation

Expectations of constants or known values:

- $\mathrm{E}[a] = a$
- $\mathrm{E}[Y \mid Y = y] = y$

# Expectation

Example: Gaussian distribution $X \sim N(\mu, \sigma^2)$

$$\mathrm{E}[X] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx$$

$$=$$

# Expectation

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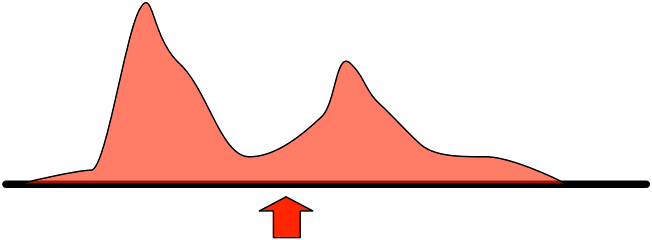
$$= \mu$$

# Expectation Intuition

- Average or outcome (might not be an event: 2.4 children)
- Center of mass



- “Fair Price” of a wager

# Expectation of die / dice

What is the expectation of the roll of die?

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**One die**

$$1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} =$$

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What is the expectation of the sum of two dice?

**Two die**

$$2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + 4 \cdot \frac{3}{36} + 5 \cdot \frac{4}{36} + 6 \cdot \frac{5}{36} + 7 \cdot \frac{6}{36} + 8 \cdot \frac{5}{36} + 9 \cdot \frac{4}{36} + 10 \cdot \frac{3}{36} + 11 \cdot \frac{2}{36} + 12 \cdot \frac{1}{36} =$$

# Expectation of die / dice

What is the expectation of the roll of die?

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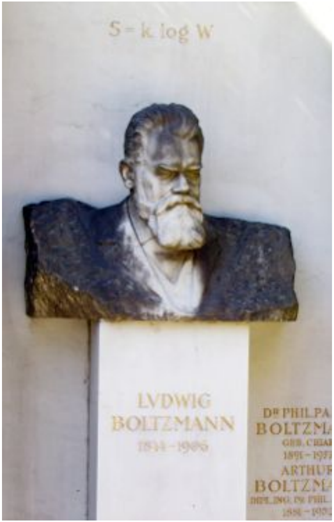
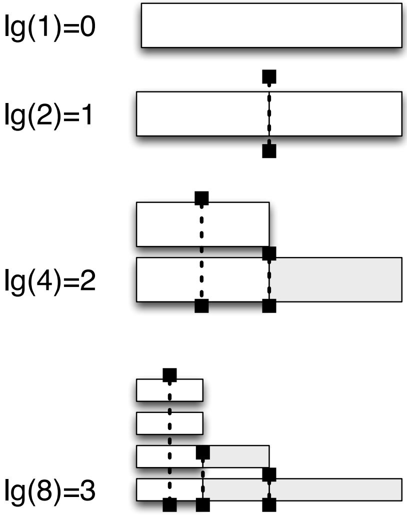
# Entropy

- Measure of disorder in a system
- In the real world, entroy in a system tends to increase
- Can also be applied to probabilities:
  - Is one (or a few) outcomes certain (low entropy)
  - Are things equiprobable (high entropy)
- In data science
  - We look for features that allow us to *reduce* entropy (decision trees)
  - All else being equal, we seek models that have *maximum* entropy (Occam's razor)



# Aside: Logarithms

- $\lg(x) = b \Leftrightarrow 2^b = x$
- Makes big numbers small
- Way to think about them: cutting a carrot



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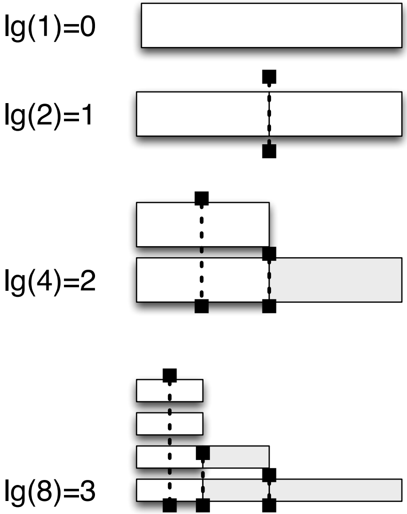

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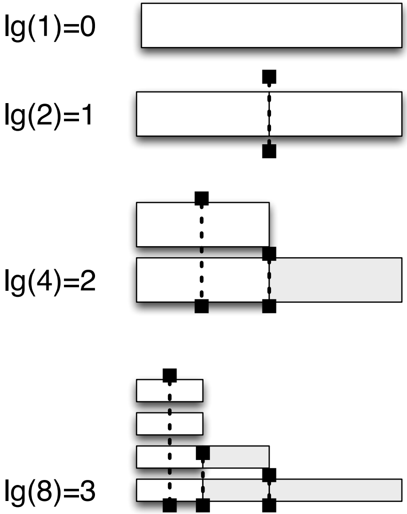
- Non-integers?



# Entropy

*Entropy* is a measure of uncertainty that is associated with the distribution of a random variable:

$$
\begin{aligned}
H(X) &= -\mathrm{E}\left[\lg(p(X))\right] \\
&= -\sum_{x} p(x)\lg(p(x)) && \text{(discrete)} \\
&= -\int_{-\infty}^{\infty} p(x)\lg(p(x))\,dx && \text{(continuous)}
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Does not account for the values of the random variable, only the spread of the distribution.

- $H(X) \geq 0$
- uniform distribution = highest entropy, point mass = lowest
- suppose $P(X=1)=p,\ P(X=0)=1-p$ and $P(Y=100)=p,\ P(Y=0)=1-p$: $X$ and $Y$ have the same entropy

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**Examples (in class)**

Entropy of one die, two dice.

# Whew!

- That's it for now
- You don't have to be an expert on this stuff (there are other classes for that)
- This is to get your feet wet and to know the concepts when you see the math

# Next Time

- Technological foundations
- Dealing with messy data
- Telling stories with data