Department of Computer Science
UNIVERSITY OF COLORADO BOULDER

Boosting

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LECTURE 10

Slides adapted from Rob Schapire

Motivating Example

Goal

Automatically categorize type of call requested by phone customer (Collect, CallingCard, PersonToPerson, etc.)

- yes I'd like to place a collect call long distance please (Collect)
- operator I need to make a call but I need to bill it to my office (ThirdNumber)
- yes I'd like to place a call on my master card please (CallingCard)
- I just called a number in sioux city and I musta rang the wrong number because I got the wrong party and I would like to have that taken off of my bill (BillingCredit)

Boosting Approach

- devise computer program for deriving rough rules of thumb
- apply procedure to subset of examples
- obtain rule of thumb
- apply to second subset of examples
- obtain second rule of thumb
- repeat T times

Details

- How to **choose** examples
- How to **combine** rules of thumb

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  concentrate on *hardest* examples (those most often misclassified by previous rules of thumb)
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  concentrate on *hardest* examples (those most often misclassified by previous rules of thumb)
- How to **combine** rules of thumb
  take (weighted) majority vote of rules of thumb

Boosting

Definition

general method of converting rough rules of thumb into highly accurate prediction rule

- assume given *weak learning algorithm* that can consistently find classifiers (rules of thumb) at least slightly better than random, say, accuracy $\geq 55\%$ (in two-class setting)
- given sufficient data, a boosting algorithm can provably construct single classifier with very high accuracy, say, 99%

Plan

Formal Description

- Training set $(x_1, y_1) \dots (x_m, y_m)$
- $y_i \in \{-1, +1\}$ is the label of instance x_i

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- For $t = 1, \dots T$:
  - Construct distribution D_t on $\{1, \dots, m\}$
  - Find weak classifier

$$h_t : \mathcal{X} \mapsto \{-1, +1\} \tag{1}$$

with small error ϵ_t on D_t:

$$\epsilon_t = \Pr_{i \sim D_t}\left[h_t(x_i) \neq y_i\right] \tag{2}$$

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  - Output final classifier H_{final}

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$$D_{t+1}(i) \propto D_t(i) \cdot \exp\{-\alpha_t y_i h_t(x_i)\} \tag{3}$$

where $\alpha_t = \frac{1}{2} \ln \left(\frac{1-\epsilon_t}{\epsilon_t} \right) > 0$

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Bigger if wrong, smaller if right

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Weight by how good the weak learner is

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- Final classifier:

$$H_{fin}(x) = \text{sign}\left(\sum_t \alpha_t h_t(x) \right) \tag{4}$$

Plan

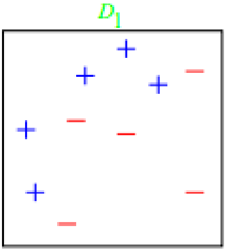
Algorithm

Example

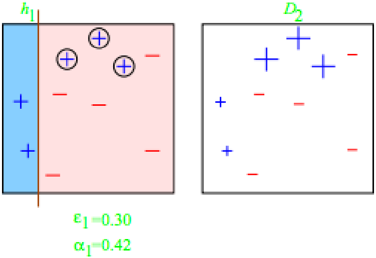
Generalization

Theoretical Analysis

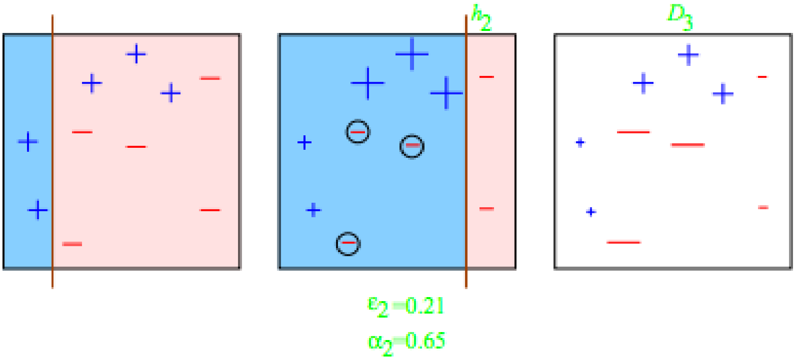
Toy Example



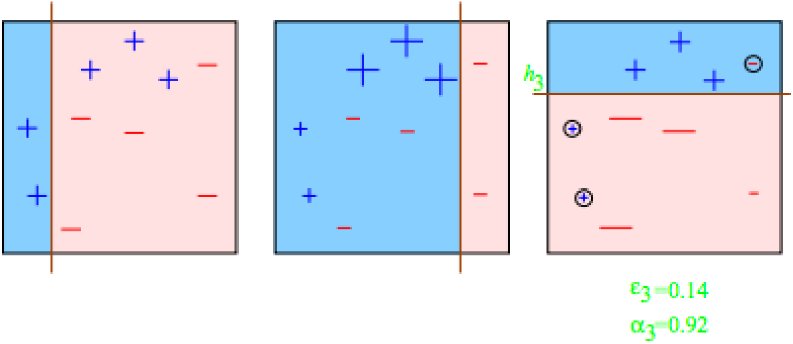
Round 1



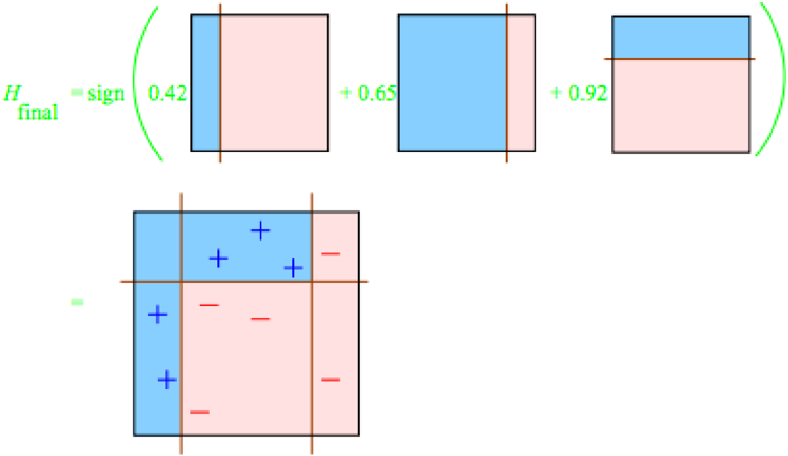
Round 2



Round 3



Final Classifier



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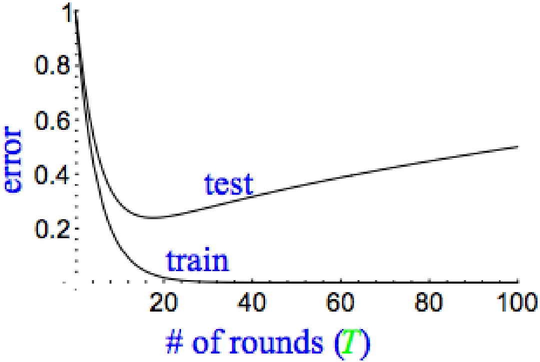
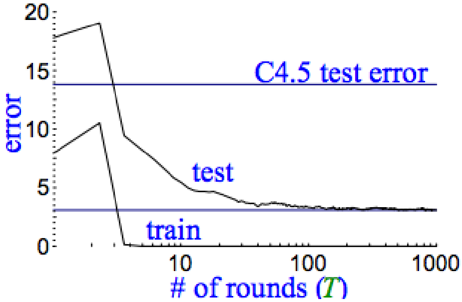


Generalization



(boosting C4.5 on "letter" dataset)

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Theoretical Analysis

Training Error

First, we can prove that the training error goes down. If we write the the error at time t as $\frac{1}{2} - \gamma_t$,

$$\hat{R}(h) \leq \exp\left\{-2 \sum_t \gamma_t^2\right\} \tag{5}$$

- If $\forall t : \gamma_t \geq \gamma > 0$, then $\hat{R}(h) \leq \exp\left\{-2\gamma^2 T\right\}$

Adaboost: do not need γ or T a priori

Training Error Proof: Preliminaries

Repeatedly expand the definition of the distribution.

$$
\begin{aligned}
D_{t+1}(i) = & \frac{D_t(i) \exp\left\{-\alpha_t y_i h_t(x_i)\right\}}{Z_t} && (6)\\
& \frac{D_{t-1}(i) \exp\left\{-\alpha_{t-1} y_i h_{t-1}(x_i)\right\} \exp\left\{-\alpha_t y_i h_t(x_i)\right\}}{Z_{t-1} Z_t} && (7)\\
& \frac{\exp\left\{-y_i \sum_{s=1}^{t} \alpha_s h_s(x_i)\right\}}{m \prod_{s=1}^{t} Z_s} && (8)
\end{aligned}
$$

Training Error Intuition

- On round t weight of examples incorrectly classified by h_t is increased
- If x_i incorrectly classified by H_T, then x_i wrong on (weighted) majority of h_t's
    - If x_i incorrectly classified by H_T, then x_i must have large weight under D_T
    - But there can't be many of them, since total weight ≤ 1

Training Error Proof: It's all about the Normalizers

$$\hat{R}(h) = \frac{1}{m} \sum_{i=1}^{m} \mathbb{1}\left[y_i g(x_i) \leq 0 \right] \tag{9}$$

$$\tag{10}$$

Definition of training error

Training Error Proof: It's all about the Normalizers

$$\hat{R}(h) = \frac{1}{m}\sum_{i=1}^{m} \mathbb{1}\left[y_i g(x_i) \leq 0\right] \tag{9}$$

$$\leq \frac{1}{m}\sum_{i=1}^{m} \exp\left\{-y_i g(x_i)\right\} \tag{10}$$

$$\tag{11}$$

$\mathbb{1}\left[u \leq 0\right] \leq \exp -u$ is true for all real u.

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Final distribution $D_{t+1}(i)$

$$D_{t+1}(i) = \frac{\exp\left\{-y_i \sum_{s=1}^{t} \alpha_s h_s(x_i)\right\}}{m \prod_{s=1}^{t} Z_s} \tag{12}$$

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$$= \frac{1}{m} \sum_{i=1}^{m} \left[m \prod_{t=1}^{T} Z_t\right] D_{T+1}(i) \tag{11}$$

$$\tag{12}$$

m's cancel, D is a distribution

Training Error Proof: It's all about the Normalizers

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$$= \prod_{t=1}^{T} Z_t \tag{12}$$

Training Error Proof: Weak Learner Errors

Single Weak Learner

$$
\begin{aligned}
Z_t &= \sum_{i=1}^{m} D_t(i) \exp\left\{-\alpha_t y_i h_t(x_i)\right\} && (13) \\
&= && (14) \\
&= && (15) \\
&= && (16)
\end{aligned}
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&= \sum_{i:\text{right}} D_t(i) \exp\left\{-\alpha_t\right\} + \sum_{i:\text{wrong}} D_t(i) \exp\left\{\alpha_t\right\} && (14) \\
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=& (1 - \epsilon_t) \exp\left\{-\alpha_t\right\} + \epsilon_t \exp\left\{\alpha_t\right\} && (15) \\
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$$= (1-\epsilon_t)\exp\{-\alpha_t\} + \epsilon_t \exp\{\alpha_t\} \tag{15}$$

$$= (1-\epsilon_t)\sqrt{\frac{\epsilon_t}{1-\epsilon_t}} + \epsilon_t\sqrt{\frac{1-\epsilon_t}{\epsilon_t}} \tag{16}$$

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Normalization Product

$$\prod_{t=1}^{T} Z_t = \prod_{t=1}^{T} 2\sqrt{\epsilon_t(1-\epsilon_t)} = \sqrt{1-4\left(\frac{1}{2}-\epsilon_t\right)^2} \tag{14}$$

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Generalization

VC Dimension

$\leq 2(d+1)(T+1)\lg\left[(T+1)e\right]$

Margin-based Analysis

AdaBoost maximizes a linear program maximizes an L_1 margin, and the weak learnability assumption requires data to be linearly separable with margin 2γ

Practical Advantages of AdaBoost

- fast
- simple and easy to program
- no parameters to tune (except T)
- flexible: can combine with any learning algorithm
- no prior knowledge needed about weak learner
- provably effective, provided can consistently find rough rules of thumb
  - shift in mind set: goal now is merely to find classifiers barely better than random guessing
- versatile
  - can use with data that is textual, numeric, discrete, etc.
  - has been extended to learning problems well beyond binary classification

Caveats

- performance of AdaBoost depends on data and weak learner
- consistent with theory, AdaBoost can fail if
- weak classifiers too complex
  - overfitting
- weak classifiers too weak ($\gamma_t \rightarrow 0$ too quickly)
  - underfitting
  - low margins $\rightarrow$ overfitting
- empirically, AdaBoost seems especially susceptible to uniform noise