Classification

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WEIGHTED MAJORITY

Slides adapted from Mohri

Beyond Binary Classification

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Beyond Binary Classification

- Before we've talked about combining weak predictor (boosting)
  - What if you have strong predictors?
- How do you make inherently binary algorithms multiclass?
- How do you answer questions like ranking?

General Online Setting

- For $t = 1$ to T:
  - Get instance $x_t \in X$
  - Predict $\hat{y}_t \in Y$
  - Get true label $y_t \in Y$
  - Incur loss $L(\hat{y}_t, y_t)$
- Classification: $Y = \{0, 1\}$, $L(y, y') = |y' - y|$
- Regression: $Y \subset \mathbb{R}, L(y, y') = (y' - y)^2$

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- **Objective**: Minimize total loss $\sum_t L(\hat{y}_t, y_t)$

Prediction with Expert Advice

- For $t = 1$ to T:
  - Get instance $x_t \in X$ and advice $a_t, i \in Y, i \in [1, N]$
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- **Objective**: Minimize regret, i.e., difference of total loss vs. best expert

$$\text{Regret}(T) = \sum_t L(\hat{y}_t, y_t) - \min_i \sum_t L(a_{t,i}, y_t) \qquad (1)$$

Mistake Bound Model

- Define the maximum number of mistakes a learning algorithm L makes to learn a concept c over any set of examples (until it's perfect).

$$M_L(c) = \max_{x_1,\ldots,x_T} |\text{mistakes}(L, c)| \tag{2}$$

- For any concept class C, this is the max over concepts c.

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- In the expert advice case, assumes some expert matches the concept (realizable)

Halving Algorithm

$H_1 \leftarrow H$;
for $t \leftarrow 1 \ldots T$ **do**
  Receive x_t;
  $\hat{y}_t \leftarrow$ Majority$(H_t, \vec{a}_t, x_t)$;
  Receive y_t;
  if $\hat{y}_t \neq y_t$ **then**
    $H_{t+1} \leftarrow \{a \in H_t : a(x_t) = y_t\}$;
return H_{T+1}

Algorithm 1: The Halving Algorithm (Mitchell, 1997)

Halving Algorithm Bound (Littlestone, 1998)

- For a finite hypothesis set

$$M_{\text{Halving}(H)} \leq \lg |H| \tag{4}$$

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$$\text{VC}(H) \leq \text{opt}(H) \leq M_{\text{Halving}(H)} \leq \lg |H| \tag{5}$$

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- For a fully shattered set, form a binary tree of mistakes with height $\text{VC}(H)$
- What about non-realizable case?

Weighted Majority (Littlestone and Warmuth, 1998)

for $i \leftarrow 1 \dots N$ **do**
  $w_{1,i} \leftarrow 1$;
for $t \leftarrow 1 \dots T$ **do**
  Receive x_t;
  $\hat{y}_t \leftarrow \mathbb{1}\left[\sum_{a_{t,i}=1} w_t \geq \sum_{a_{t,i}=0} w_t\right]$;
  Receive y_t;
  if $\hat{y}_t \neq y_t$ **then**
    for $i \leftarrow 1 \dots N$ **do**
      if $a_{t,i} \neq y_t$ **then**
        $w_{t+1,i} \leftarrow \beta w_{t,i}$;
      else
        $w_{t+1,i} \leftarrow w_{t,i}$
return w_{T+1}

- Weights for every expert
- Classifications in favor of side with higher total weight ($y \in \{0,1\}$)
- Experts that are wrong get their weights decreased ($\beta \in [0,1]$)
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Weighted Majority

- Let m_t be the number of mistakes made by WM until time t
- Let m_t^* be the best expert's mistakes until time t
- N is the number of experts

$$m_t \leq \frac{\log N + m_t^* \log \frac{1}{\beta}}{\log \frac{2}{1+\beta}} \tag{6}$$

- Thus, mistake bound is $O(\log N)$ plus the best expert
- Halving algorithm $\beta = 0$

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Weights are nonnegative, so $\sum_i w_{t,i} \geq w_{t,i}$

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Each error multiplicatively reduces weight by β

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- After m_T mistakes after T rounds

$$\Phi_T \leq \left[\frac{1+\beta}{2} \right]^{m_T} N \tag{11}$$

Weighted Majority Proof

- Put the two inequalities together, using the best expert

$$\beta^{m_T^*} \leq \Phi_T \leq \left[\frac{1+\beta}{2}\right]^{m_T} N \tag{12}$$

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- Solve for m_T

$$m_T \leq \frac{\log N + m_T^* \log \frac{1}{\beta}}{\log\left[\frac{2}{1+\beta}\right]} \tag{14}$$

Weighted Majority Recap

- Simple algorithm
- No harsh assumptions (non-realizable)
- Depends on best learner
- Downside: Takes a long time to do well in worst case (but okay in practice)
- Solution: Randomization