Dirichlet Processes

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INTRODUCTION

Clustering as Probabilistic Inference

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- There are several latent variables:
  - Means
  - Assignments
  - (Variances)

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  - Means
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  - (Variances)
- Before, we were doing EM
- Today, new models and new methods

Nonparametric Clustering

- What if the number of clusters is not fixed?
- Nonparametric: can grow if data need it
- Probabilistic distribution over number of clusters

Dirichlet Process

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- You can then draw observations from $x \sim \mathrm{DP}(\alpha, G)$.

Defining a DP

- Break off sticks

$$V_1, V_2, \cdots \sim_{\text{iid}} \text{Beta}(1, \alpha) \tag{1}$$

$$C_k \equiv V_k \prod_{j=1}^{k-1} (1 - V_j) \tag{2}$$

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- Draw atoms

$$\Phi_1, \Phi_2, \ldots \sim_{\text{iid}} G$$

- Merge into complete distribution

$$\Theta = \sum_{k \in \mathbb{N}} C_k \delta_{\Phi_k}$$

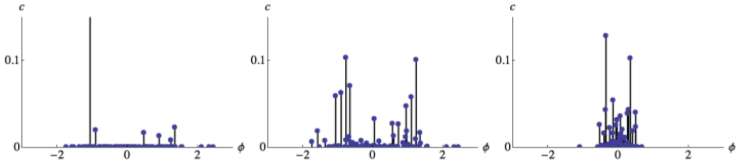
Properties of a DPMM

- Expected value is the same as base distribution

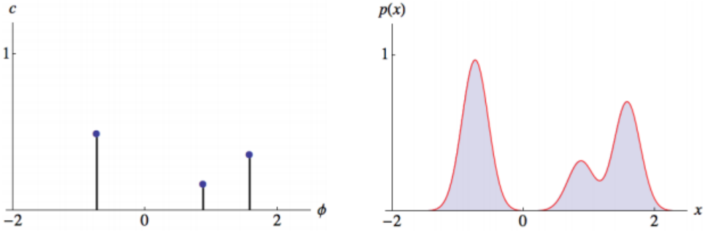
$$\mathbb{E}_{\text{DP}(\alpha,G)}[x] = \mathbb{E}_G[x] \tag{3}$$

- As $\alpha \rightarrow \infty$, $\text{DP}(\alpha, G) = G$
- Number of components unbounded
- Impossible to represent fully on computer (truncation)
- You can nest DPs

Effect of scaling parameter α

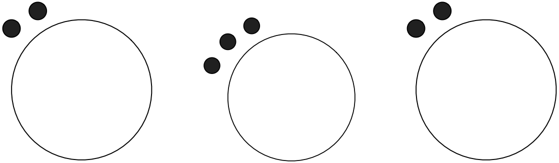


DP as mixture Model



The Chinese Restaurant as a Distribution

To generate an observation, you first sit down at a table. You sit down at a table proportional to the number of people sitting at the table.



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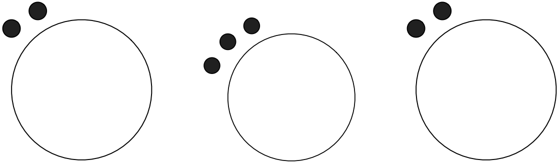

$\frac{2}{7}$ $\frac{3}{7}$ $\frac{2}{7}$

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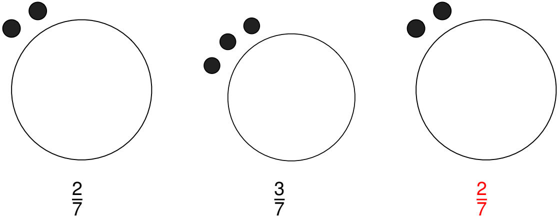

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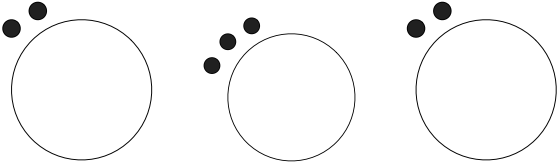

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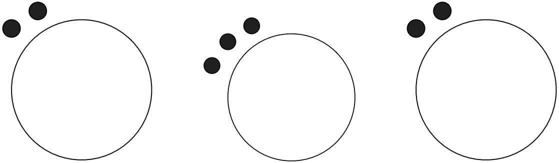

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But this is just Maximum Likelihood

Why are we talking about Chinese Restaurants?

Always can squeeze in one more table ...

- The *posterior* of a DP is CRP
- A new observation has a new table / cluster with probability proportional to α
- But this must be balanced against the probability of an observation *given a cluster*

$$\Theta = \sum_{k \in \mathbb{N}} C_k \delta_{\Phi_k}$$

Gibbs Sampling

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- This provides a mean for each cluster
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Gibbs Sampling

- We want to know $\vec{z}$
- Compute $p(z_i \,|\, z_1 \dots z_{i-1}, z_{i+1}, \dots z_m, x, \alpha, G)$
- Update z_i by sampling from that distribution
- Keep going ...

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Notation

$$p(z_i = k \,|\, z_{-i}) \equiv p(z_i \,|\, z_1 \dots z_{i-1}, z_{i+1}, \dots z_m) \quad (4)$$

Gibbs Sampling for DPMM

$$p(z_i = k \,|\, \vec{z}_{-i}, \vec{x}, \{\theta_k\}, \alpha) \tag{5}$$

(6)

Gibbs Sampling for DPMM

$$p(z_i = k \,|\, \vec{z}_{-i}, \vec{x}, \{\theta_k\}, \alpha) \tag{5}$$

$$= p(z_i = k \,|\, \vec{z}_{-i}, x_i, \vec{x}, \theta_k, \alpha) \tag{6}$$

$$\tag{7}$$

Dropping irrelevant terms

Gibbs Sampling for DPMM

$$
\begin{aligned}
& p(z_i = k \,|\, \vec{z}_{-i}, \vec{x}, \{\theta_k\}, \alpha) && (5) \\
= & p(z_i = k \,|\, \vec{z}_{-i}, x_i, \vec{x}, \theta_k, \alpha) && (6) \\
= & p(z_i = k \,|\, \vec{z}_{-i}, \alpha) p(x_i \,|\, \theta_k, \vec{x}) && (7) \\
& && (8)
\end{aligned}
$$

Chain rule

Gibbs Sampling for DPMM

$$
\begin{aligned}
& p(z_i = k \,|\, \vec{z}_{-i}, \vec{x}, \{\theta_k\}, \alpha) && (5)\\
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=& p(z_i = k \,|\, \vec{z}_{-i}, \alpha) p(x_i \,|\, \theta_k, \vec{x}) && (7)\\
=& \begin{cases} \left(\frac{n_k}{n_\cdot + \alpha}\right) \int_\theta p(x_i \,|\, \theta) p(\theta \,|\, G, \vec{x}) & \text{existing} \\ \frac{\alpha}{n_\cdot + \alpha} \int_\theta p(x_i \,|\, \theta) p(\theta \,|\, G) & \text{new} \end{cases} && (8)\\
& && (9)
\end{aligned}
$$

Applying CRP

Gibbs Sampling for DPMM

$$p(z_i = k \,|\, \vec{z}_{-i}, \vec{x}, \{\theta_k\}, \alpha) \tag{5}$$

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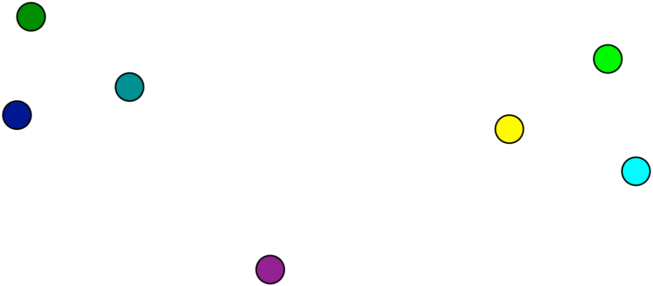
$$= \begin{cases} \left(\frac{n_k}{n_\cdot + \alpha}\right) \mathcal{N}\left(x, \frac{n\bar{x}}{n+1}, \mathbb{1}\right) & \text{existing} \\ \frac{\alpha}{n_\cdot + \alpha} \mathcal{N}(x, 0, \mathbb{1}) & \text{new} \end{cases} \tag{9}$$

Scary integrals assuming G is normal distribution with mean zero and unit variance. (Derived in optional reading.)

Algorithm for Gibbs Sampling

1. Random initial assignment to clusters
2. For iteration i:
   2.1 "Unassign" observation n
   2.2 Choose new cluster for that observation

Toy Example



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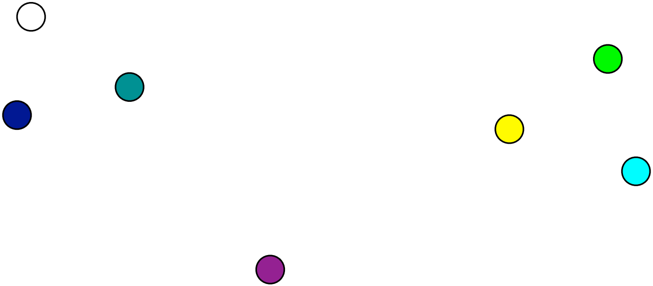

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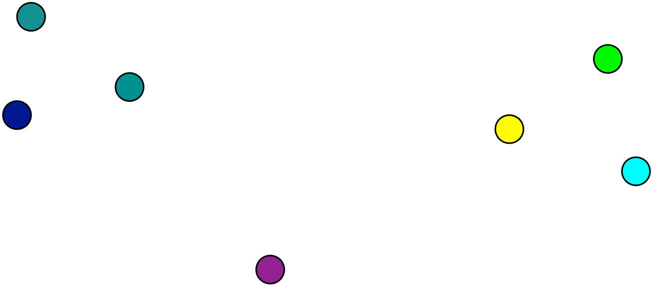

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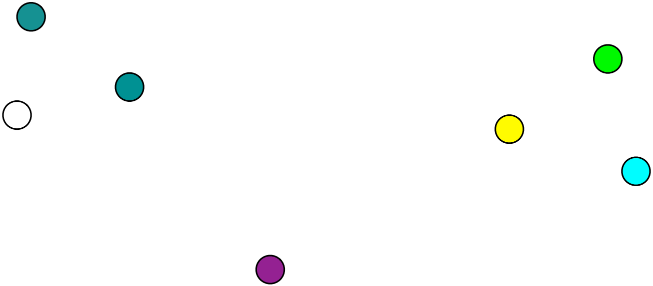

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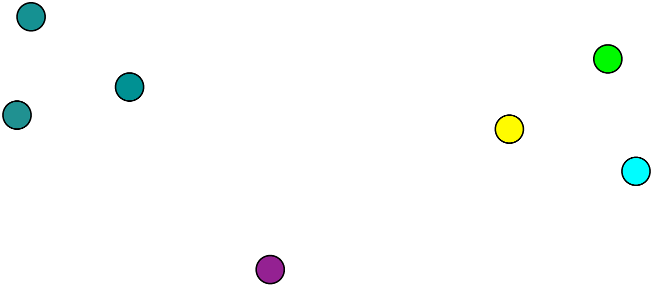

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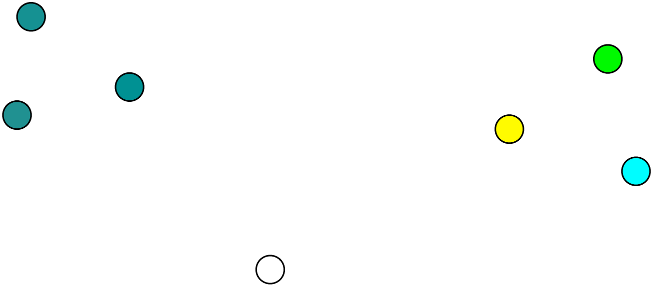

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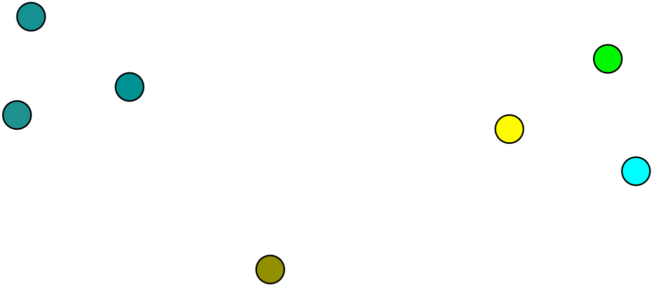

New cluster created!

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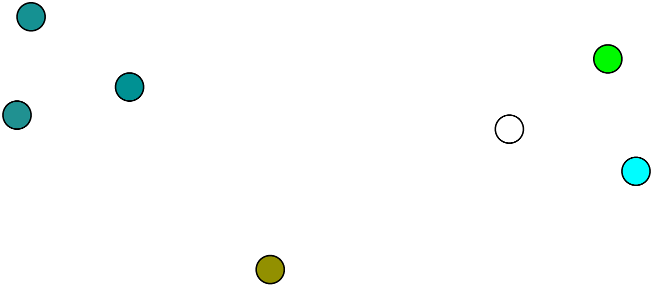

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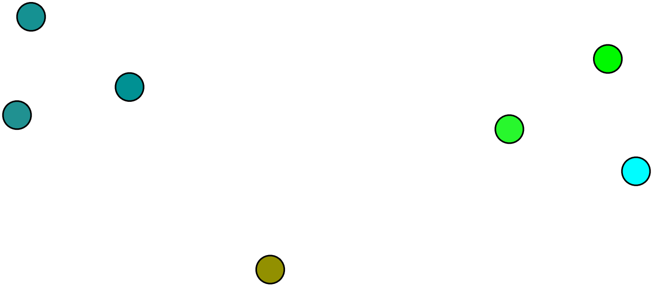

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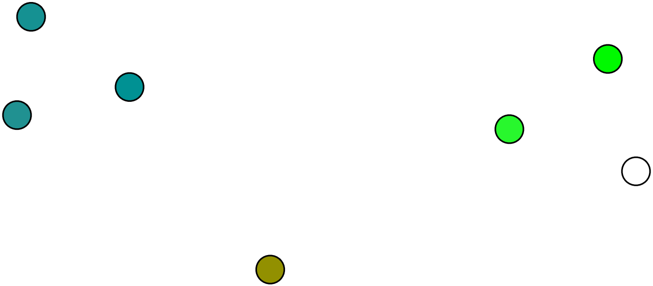

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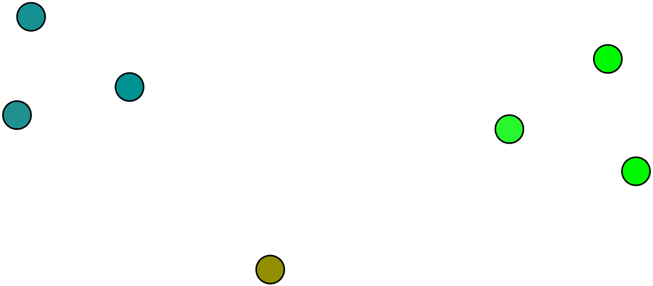

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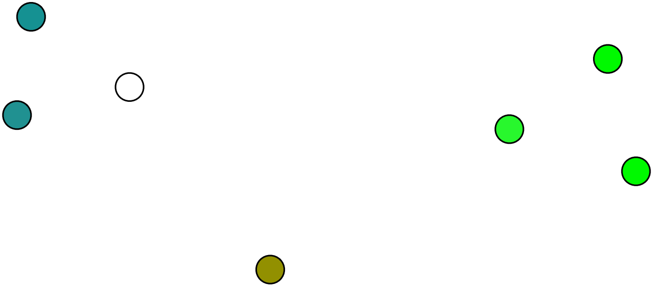

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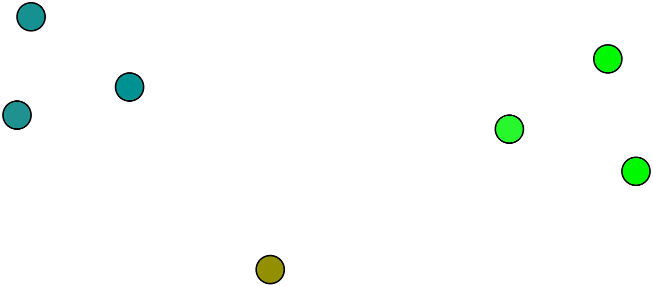

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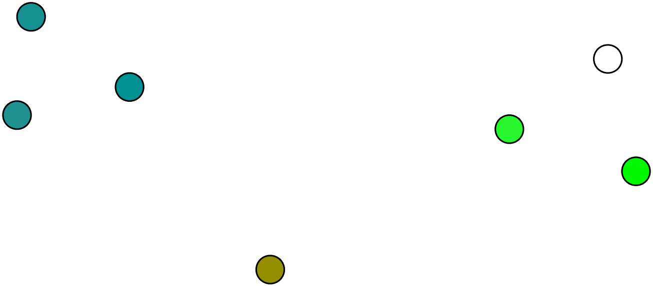

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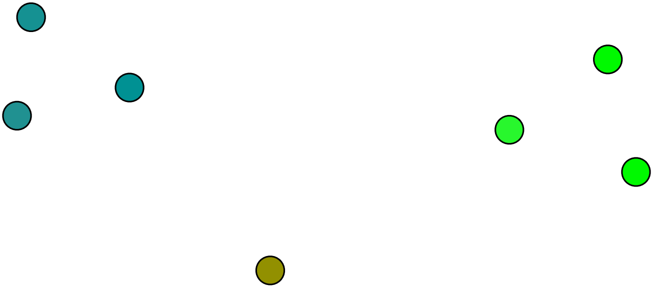

And repeat . . .

Differences between EM and Gibbs

- Gibbs often faster to implement
- EM easier to diagnose convergence
- EM can be parallelized
- Gibbs is more widely applicable

In class and next week

- Walking through DPMM clustering
- Clustering discrete data with more than one cluster per observation