Introduction to Machine Learning

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RADEMACHER COMPLEXITY

Content Questions

Administrivia Questions

Single Hypothesis

What is the Rademacher complexity of a hypothesis set reduced to a single hypothesis?

$$ \tag{1} $$

Single Hypothesis

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$$\mathscr{R}_m(H) = \mathbb{E}_{S \sim D^m, \sigma}\left[\sup_{h \in H} \frac{1}{m} \sum_{i=1}^{m} \sigma_i h(z_i)\right] \tag{1}$$

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$$(5)$$

Rademacher Identity 1

Prove

$$\mathscr{R}_m(\alpha H) = |\alpha|\mathscr{R}_m(H)$$

If $\alpha \geq 0$

If $\alpha < 0$

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$$\alpha \sup_{h \in H} \sum_{i=1}^{m} \sigma_i h(x_i) \tag{8}$$

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$$\sup_{h \in \alpha H} \sum_{i=1}^{m} \sigma_i h(x_i) = \quad (9)$$

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Since σ_i and $-\sigma$ have the same distribution, $\mathscr{R}_m(\alpha H) = |\alpha| \mathscr{R}_m(H)$

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(15)

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VC Dimension

To show VC dimension of a set of points

- Show that **a** set of d *can* be shattered
- Show that **no** set of $d+1$ can be shattered

Axis Aligned Rectangles

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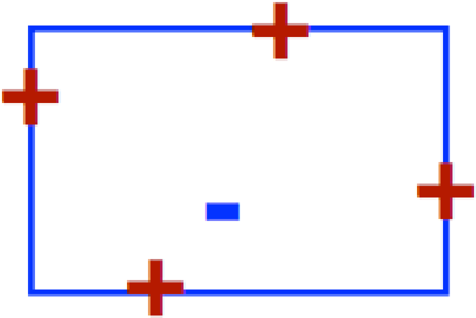

Hyperplanes

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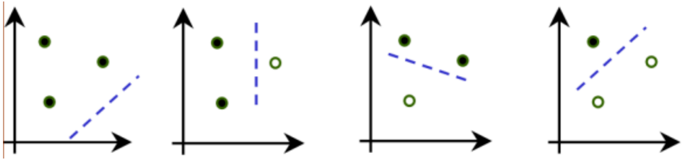

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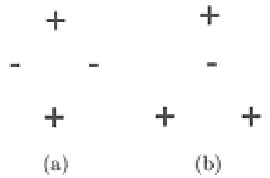

Figure 3.2 Unrealizable dichotomies for four points using hyperplanes in $\mathbb{R}^2$. (a) All four points lie on the convex hull. (b) Three points lie on the convex hull while the remaining point is interior.

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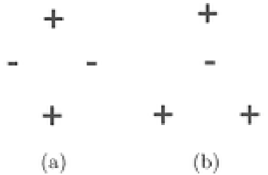

Figure 3.2 Unrealizable dichotomies for four points using hyperplanes in $\mathbb{R}^2$. (a) All four points lie on the convex hull. (b) Three points lie on the convex hull while the remaining point is interior.

In general, the VC dimension of d-dimensional hyperplanes is $d + 1$

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- To shatter a set, it means that every point can take on a different binary label $h(x)$
- If a set has d points, there are 2^d ways to do that
- Each configuration requires a different hypothesis
- Solving for the number of hypotheses gives $\lg |H|$

Next time

- Getting more practical
- SVMs
- Excellent theoretical properties