# Introduction to Machine Learning

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RADEMACHER COMPLEXITY

Slides adapted from Rob Schapire

## Recap

- Rademacher complexity provides nice guarantees

$$R(h) \leq \hat{R}(h) + \mathscr{R}_m(H) + \mathcal{O}\left(\sqrt{\frac{\log \frac{1}{\delta}}{2m}}\right) \tag{1}$$

- But in practice hard to compute for real hypothesis classes
- Is there a relationship with simpler combinatorial measures?

## Growth Function

Define the **growth function** $\Pi_H : \mathbb{N} \rightarrow \mathbb{N}$ for a hypothesis set $H$ as:

$$\forall m \in \mathbb{N}, \Pi_H(m) \equiv \max_{\{x_1,\dots,x_m\} \in X} \left| \{ (h(x_1), \dots, h(x_m) : h \in H\} \right| \tag{2}$$

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i.e., the number of ways $m$ points can be classified using $H$.

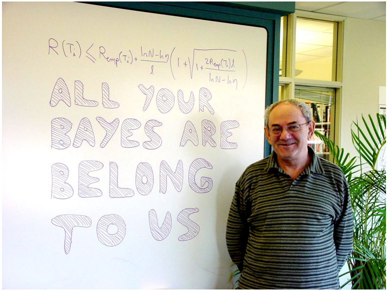
## Rademacher Complexity vs. Growth Function

If $G$ is a function taking values in $\{-1, +1\}$, then

$$\mathscr{R}_m(G) \leq \sqrt{\frac{2 \ln \Pi_G(m)}{m}} \tag{3}$$

Uses Masart's lemma

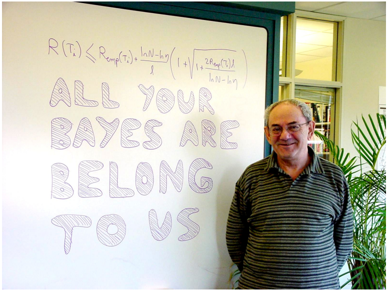
# Vapnik-Chervonenkis Dimension





$$\mathrm{VC}(H) \equiv \max\left\{m : \Pi_H(m) = 2^m\right\} \tag{4}$$

## Vapnik-Chervonenkis Dimension





$$\text{VC}(H) \equiv \max\left\{m : \Pi_H(m) = 2^m\right\} \tag{4}$$

The size of the largest set that can be fully shattered by $H$.

## VC Dimension for Hypotheses

- Need upper and lower bounds
- Lower bound: example
- Upper bound: Prove that no set of $d+1$ points can be shattered by $H$ (harder)

## Intervals

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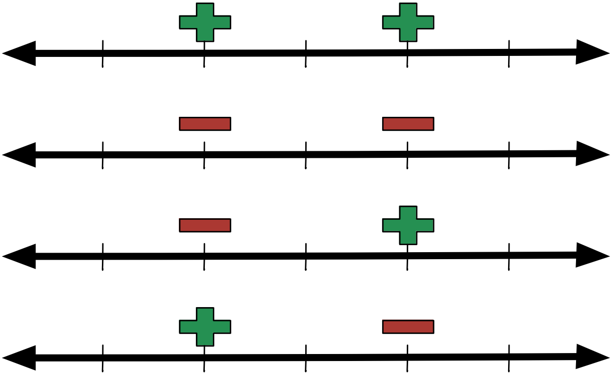

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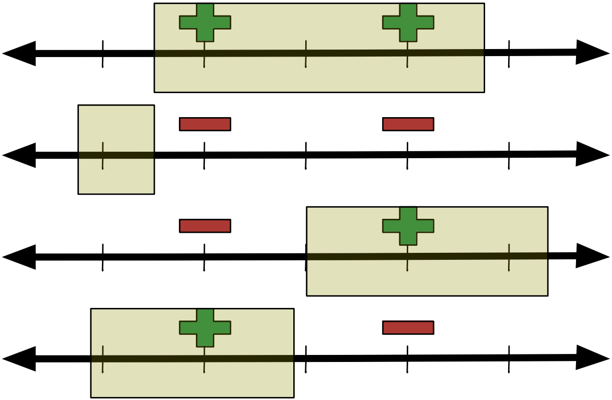

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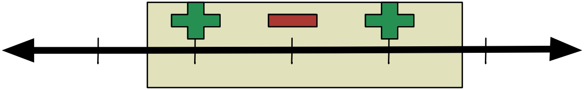

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- What about three points?
- **No set** of three points can be shattered
- Thus, VC dimension of intervals is 2

## Sine Functions

- Consider hypothesis that classifies points on a line as either being above or below a sine wave

$$\{t \longrightarrow \sin(\omega x) : \omega \in \mathbb{R}\} \tag{5}$$

- Can you shatter three points?

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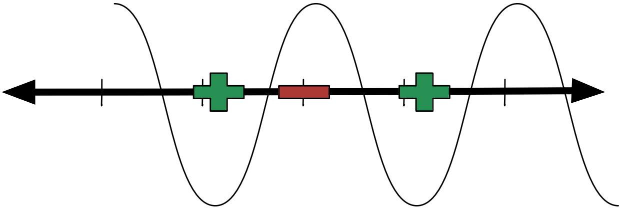

## Sine Functions

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- Can you shatter four points?

## Sine Functions

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- How many points can you shatter?

### Sine Functions

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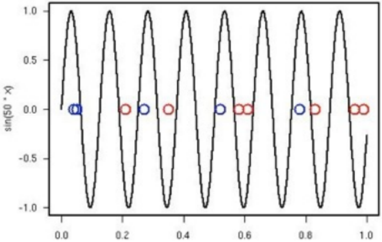
$$\{t \longrightarrow \sin(\omega x) : \omega \in \mathbb{R}\} \tag{5}$$

- Thus, VC dim of sine on line is $\infty$



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VC dimension obviously encodes the complexity of a hypothesis class, but we want to connect that to Rademacher complexity and the growth function so we can prove generalization bounds.

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### Theorem

**Sauer's Lemma** *Let $H$ be a hypothesis set with VC dimension $d$. Then* $\forall m \in \mathbb{N}$

$$\Pi_H(m) \leq \sum_{i=0}^{d} \binom{m}{i} \equiv \Phi_d(m) \tag{6}$$

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This is good because the sum when multiplied out becomes $\binom{m}{i} = \frac{m \cdot (m-1) \ldots}{i!} = \mathcal{O}\left(m^d\right)$. When we plug this into the learning error limits: $\log(\Pi_H(2m)) = \log(\mathcal{O}\left(m^d\right)) = \mathcal{O}\left(d \log m\right)$.

## Proof of Sauer's Lemma

Prelim:

$$\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1} \qquad \text{This comes from Pascal's Triangle}$$

$$\binom{m}{k} = 0 \quad \text{if} \begin{cases} k < 0 \\ k > m \end{cases} \qquad \text{This convention is consistent with Pascal's Triangle}$$

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We'll proceed by induction. Our two base cases are:

- If $m = 0$, $\Pi_H(m) = 1$. You have no data, so there's only one (degenerate) labeling
- If $d = 0$, $\Pi_H(m) = 1$. If you can't even shatter a single point, then it's a fixed function

## Induction Step

Assume that it holds for all $m'$, $d'$ for which $m' + d' < m + d$. We are given $H$, $|S| = m$, $S = \langle x_1, \ldots, x_m \rangle$, and $d$ is the VC dimension of $H$.

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### Build two new hypothesis spaces

| $\mathcal{H}$ | | | | | | | $\mathcal{H}_1$ | | | | | | $\mathcal{H}_2$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\mathbf{x}_1, \ldots, \mathbf{x}_m$ | | | | | | | $\mathbf{x}_1, \ldots, \mathbf{x}_{m-1}$ | | | | | | $\mathbf{x}_1, \ldots, \mathbf{x}_{m-1}$ | | | |
| h1 | 0 | 1 | 1 | 0 | 0 | $\rightarrow$ | h1 | 0 | 1 | 1 | 0 | $\rightarrow$ | h1 | 0 | 1 | 1 | 0 |
| h2 | 0 | 1 | 1 | 0 | 1 | $\nearrow$ | | | | | | | | | | | |
| h3 | 0 | 1 | 1 | 1 | 0 | $\rightarrow$ | h3 | 0 | 1 | 1 | 1 | | | | | | |
| h4 | 1 | 0 | 0 | 1 | 0 | $\rightarrow$ | h4 | 1 | 0 | 0 | 1 | $\rightarrow$ | h4 | 1 | 0 | 0 | 1 |
| h5 | 1 | 0 | 0 | 1 | 1 | $\nearrow$ | | | | | | | | | | | |
| h6 | 1 | 1 | 0 | 0 | 1 | $\rightarrow$ | h6 | 1 | 1 | 0 | 0 | | | | | | |

Encodes where the extended set has differences on the first $m$ points.

## What is VC dimension of $H_1$ and $H_2$?

- If a set is shattered by $H_1$, then it is also shattered by $H$

$$\text{VC-dim}(H_1) \leq \text{VC-dim}(H) = d \tag{7}$$

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- If a set $T$ is shattered by $H_2$, then $T \cap \{x_m\}$ is shattered by $H$ since there will be two hypotheses in $H$ for every element of $H_2$ by adding $x_m$

$$\text{VC-dim}(H_2) \leq d - 1 \tag{8}$$

## Bounding Growth Function

$$|\Pi_H(S)| = |H_1| + |H_2| \tag{9}$$

$$\leq \sum_{i=0}^{d} \binom{m-1}{i} + \sum_{i=0}^{d-1} \binom{m-1}{i} \tag{10}$$

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We can rewrite this as $\sum_{i=0}^{d} \binom{m-1}{i-1}$ because $\binom{x}{-1} = 0$.

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\begin{aligned}
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Pascal's Triangle

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$$= \sum_{i=0}^{d} \left[ \binom{m-1}{i} + \binom{m-1}{i-1} \right] \tag{11}$$

$$= \sum_{i=0}^{d} \binom{m}{i} \tag{12}$$

$$= \Phi_d(m) \tag{13}$$

## Wait a minute . . .

Is this combinatorial expression really $\mathcal{O}\left(m^d\right)$?

$$
\begin{aligned}
\sum_{i=0}^{d} \binom{m}{i} &\leq \sum_{i=0}^{d} \binom{m}{i} \left(\frac{m}{d}\right)^{d-i} \\
&\leq \sum_{i=0}^{m} \binom{m}{i} \left(\frac{m}{d}\right)^{d-i} \\
&= \left(\frac{m}{d}\right)^{d} \sum_{i=0}^{m} \binom{m}{i} \left(\frac{d}{m}\right)^{i} \\
&= \left(\frac{m}{d}\right)^{d} \left(1 + \frac{d}{m}\right)^{m} \leq \left(\frac{m}{d}\right)^{d} e^{d}.
\end{aligned}
$$

## Generalization Bounds

Combining our previous generalization results with Sauer's lemma, we have that for a hypothesis class $H$ with VC dimension $d$, for any $\delta > 0$ with probability at least $1-\delta$, for any $h \in H$,

$$R(h) \leq \hat{R}(h) + \sqrt{\frac{2d \log \frac{em}{d}}{m}} + \sqrt{\frac{\log \frac{1}{\delta}}{2m}} \tag{14}$$

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- And works well in practice . . . Support Vector Machines
- In class: more VC dimension examples