Computational Methods
CMSC/AMSC/MAPL 460

Fourier transform

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Fourier Methods

- Last class
  - Introduced the Fourier basis
  - Showed why it might be useful
  - Introduced the notion of a Fourier Matrix
  - Introduced the Danielson Lanczos Lemma
  - Introduced the FFT algorithm
- This class
  - Detailed consideration of the FFT algorithm
  - Inverse FFT
  - Application to polynomial multiplication

Complex Notation

For f, periodic with period p

Fourier transform $f(t) \rightarrow F[k]$

$$F[k] = 1/p \int_0^p f(t)\, e^{-2\pi i k t/p}\, dt$$

$$= 1/p \int_0^p f(t) \cos(2\pi k t/p)\, dt$$

$$- i/p \int_0^p f(t) \sin(2\pi k t/p)\, dt$$

Inverse Fourier transform $F[k] \rightarrow f(t)$

$$f(t) = \sum_{k \text{ in } Z} F[k]\, e^{2\pi i k t/p}$$

Sampling

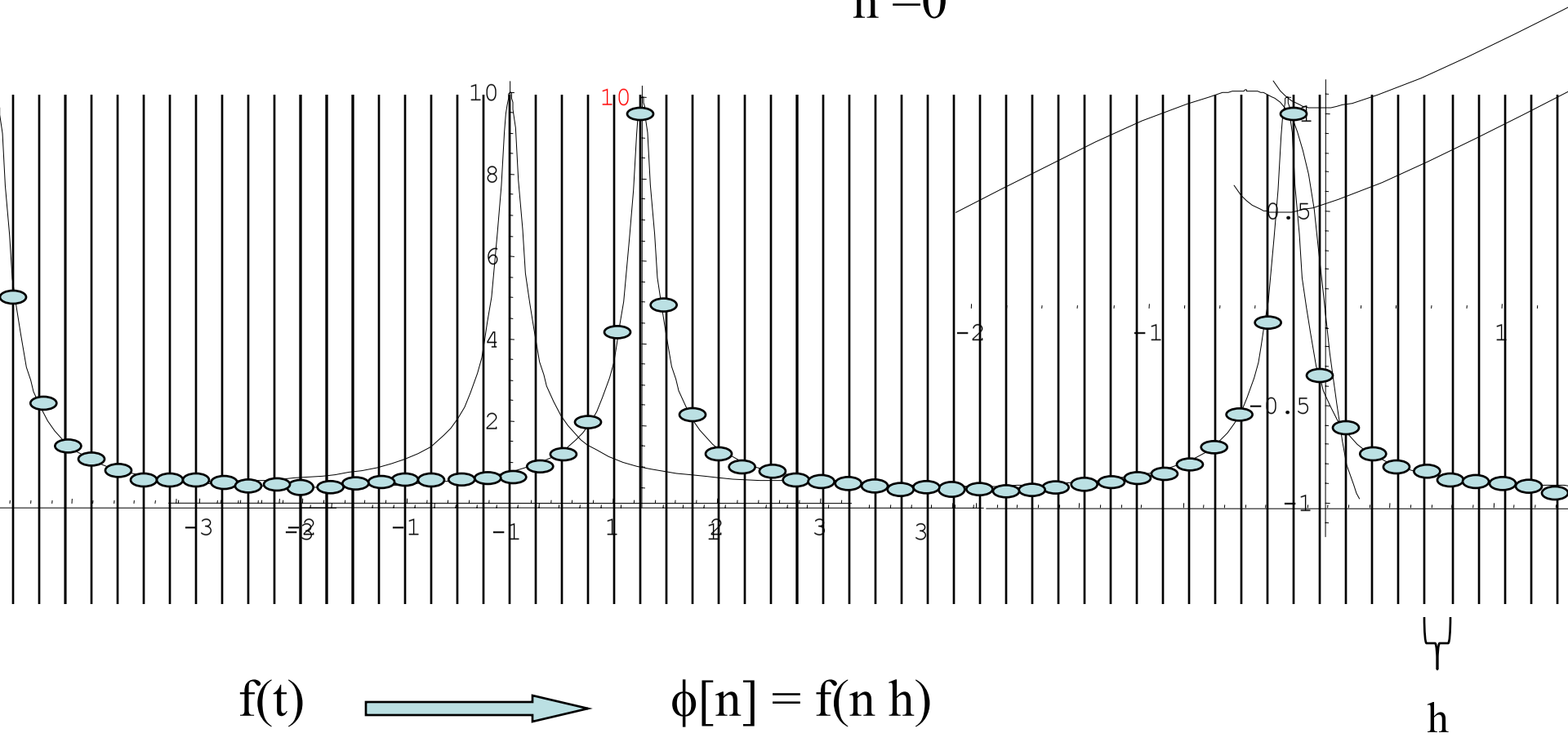
Fourier representations work just fine with sampled data

Simple connection to Fourier of the continuous function it came form

Familiar example: Digital Audio

DFT

$$\int_0^p f(t)\; e^{-2\pi i k t/p}\, dt \longrightarrow \sum_{n=0}^{N-1} \phi[n]\; e^{-2\pi i k n h/p}$$



$f(t) \longrightarrow \phi[n] = f(n\,h)$

h

DFT and its inverse for periodic discrete data

$$\Phi[k] = \sum_{n=0}^{N-1} \phi[n]\ e^{-2\pi i k n h/p} \qquad p = N h$$

$$= \sum_{n=0}^{N-1} \phi[n]\ e^{-2\pi i k n/N}$$

This is automatically periodic in k with period N

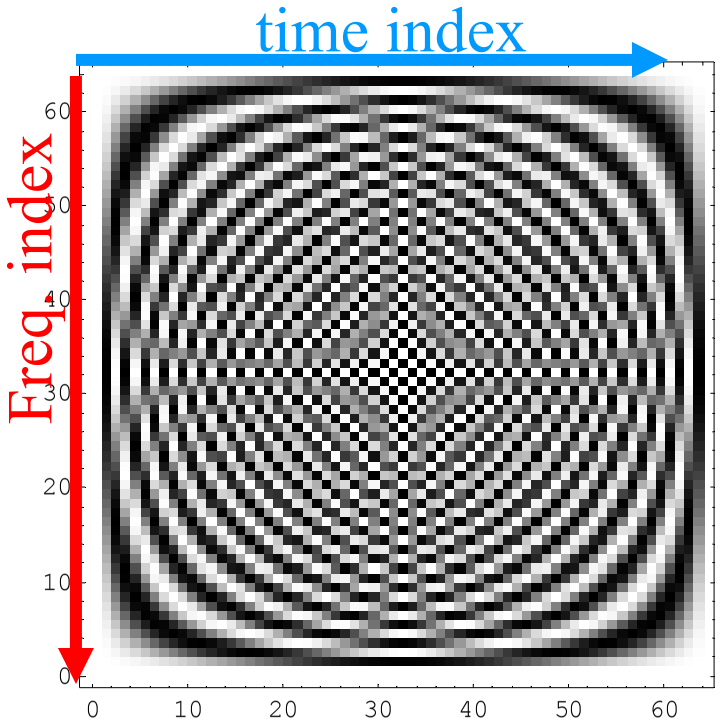
Inverse is like Fourier series, but with only p terms

Discrete time Numerical Fourier Analysis

DFT is really just a matrix multiplication!

$$F[m] = 1/N \sum_{k=0}^{N-1} e^{-2\pi i k m/N} f[k]$$

$$\begin{pmatrix} F[0] \\ F[1] \\ F[2] \\ \vdots \\ F[N-1] \end{pmatrix} =$$



$$\begin{pmatrix} f[0] \\ f[1] \\ f[2] \\ \vdots \\ f[N-1] \end{pmatrix}$$

$$F = F_N f$$

Fourier Matrices

A Fourier matrix of order n is defined as the following

$$F_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \omega_n^2 & \cdots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \cdots & \omega_n^{2(n-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \cdots & \omega_n^{(n-1)(n-1)} \end{bmatrix},$$

where

$$\omega_n = e^{-\frac{2\pi i}{n}},$$

is an nth root of unity.

$i = \sqrt{-1}$

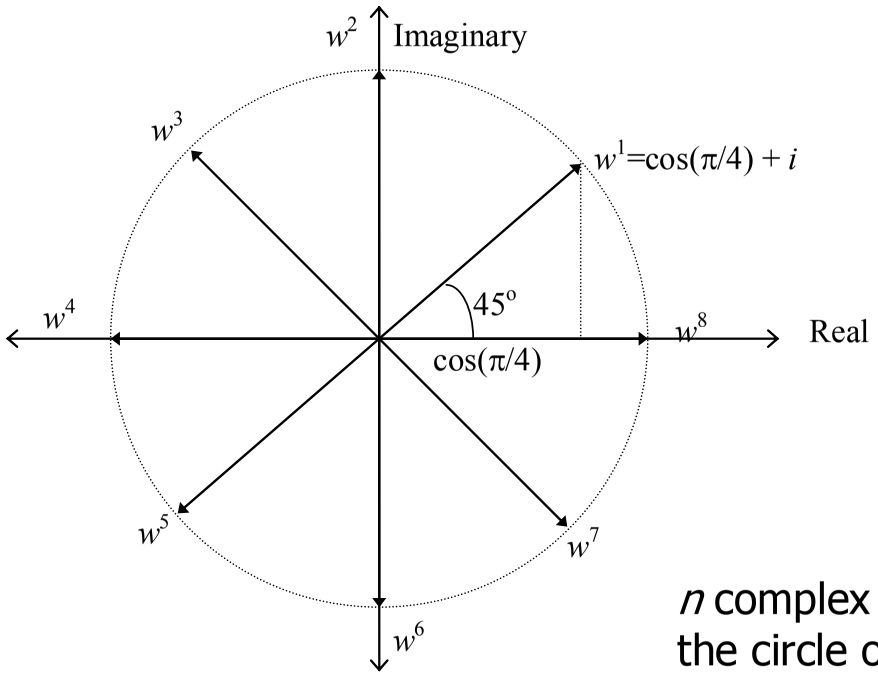
Primitive Roots of Unity

A number ω is a ***primitive n-th root of unity***, for n>1, if

$\omega^n = 1$

The numbers 1, ω, ω^2, …, ω^{n-1} are all distinct

- Example: The complex number $e^{2pi/n}$ is a primitive n-th root of unity, where



Check: if properties are satisfied

1. $\omega^1 = e^{\frac{2\pi i}{n}} \neq 1$
2. $\omega^n = \left(e^{\frac{2\pi i}{n}} \right)^n = e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1$
3. $S = \sum_{p=0}^{n-1} \omega^{jp} = \omega^0 + \omega^j + \omega^{2j} + \omega^{3j} + \ldots\ldots + \omega^{j(n-1)} = 0$

n complex roots of unity equally spaced around the circle of unit radius centered at the origin of the complex plane.

Roots of Unity: Properties

- Property 1: Let ω be the principal n^{th} root of unity. If $n > 0$, then $\omega^{n/2} = -1$.
  - Proof: $\omega = e^{2\pi i / n} \Rightarrow \omega^{n/2} = e^{\pi i} = -1$. (Euler's formula)
  - **Reflective Property:**
  - Corollary: $\omega^{k+n/2} = -\omega^{k}$.

- Property 2: Let $n > 0$ be even, and let ω and ν be the principal n^{th} and $(n/2)^{th}$ roots of unity. Then $(\omega^{k})^{2} = \nu^{k}$.
  - Proof: $(\omega^{k})^{2} = e^{(2k)2\pi i / n} = e^{(k)\,2\pi i / (n/2)} = \nu^{k}$.
  - **Reduction Property:** If ω is a primitive (2n)-th root of unity, then ω^{2} is a primitive n-th root of unity.

- L3: Let $n > 0$ be even. Then, the squares of the n complex n^{th} roots of unity are the n/2 complex $(n/2)^{th}$ roots of unity.
  - Proof: If we square all of the n^{th} roots of unity, then each $(n/2)^{th}$ root is obtained exactly twice since:
    - L1 $\Rightarrow$ $\omega^{k+n/2} = -\omega^{k}$
    - thus, $(\omega^{k+n/2})^{2} = (\omega^{k})^{2}$
    - L2 $\Rightarrow$ both of these = ν^{k}
    - $\omega^{k+n/2}$ and ω^{k} have the same square
- **Inverse Property:** If ω is a primitive root of unity, then $\omega^{-1}=\omega^{n-1}$
  - Proof: $\omega\omega^{n-1}=\omega^{n}=1$

Fast Fourier Transform

- Presented by Cooley and Tukey in 1965, but invented several times, including by Gauss (1809) and Danielson & Lanczos (1948)
- Danielson Lanczos lemma

$$
\begin{aligned}
F_k &= \sum_{j=0}^{N-1} e^{2\pi ijk/N} f_j \\
&= \sum_{j=0}^{N/2-1} e^{2\pi ik(2j)/N} f_{2j} + \sum_{j=0}^{N/2-1} e^{2\pi ik(2j+1)/N} f_{2j+1} \\
&= \sum_{j=0}^{N/2-1} e^{2\pi ikj/(N/2)} f_{2j} + W^k \sum_{j=0}^{N/2-1} e^{2\pi ikj/(N/2)} f_{2j+1} \\
&= F_k^e + W^k \, F_k^o
\end{aligned}
$$

- So far we have seen what happens on the right hand side
- How about the left hand side?
- When we split the sums in two we have two sets of sums with *N/2* quantities for *N* points.
- So the complexity is $N^2/2 + N^2/2 = N^2$
- So there is no improvement
- Need to reduce the sums on the right hand side as well
  - We need to reduce the number of sums computed from 2N to a lower number
  - Notice that the values corresponding to *k* and *k+N/2* will be the same.
  - The transforms F_e^k and F_o^k are periodic in *k* with length *N/2*.
  - So we need only compute half of them!

FFT

- So DFT of order *N* can be expressed as sum of two DFTs of order *N/2 evaluated at N/2* points
- Does this improve the complexity?
- Yes $(N/2)^2+(N/2)^2 = N^2/2 < N^2$
- But we are not done ….
- Can apply the lemma recursively

$$F_k^e = F_k^{ee} + W^k F_k^{eo}, \quad F_k^o = F_k^{oe} + W^k F_k^{oo},$$

- Finally we have a set of one point transforms
- One point transform is identity

$$F_k^{eoeeoeo\cdots oee} = f_n$$

FFT Algorithm

FFT ($n, a_0, a_1, a_2, \ldots, a_{n-1}$)

```
if (n == 1)                    // n is a power of 2
    return a_0

ω ← e^(2π i / n)
(e_0, e_1, e_2, ..., e_{n/2-1}) ← FFT(n/2, a_0, a_2, a_4, ..., a_{n-2})
(d_0, d_1, d_2, ..., d_{n/2-1}) ← FFT(n/2, a_1, a_3, a_5, ..., a_{n-1})

for k = 0 to n/2 - 1
      y_k ← e_k + ω^k d_k
  y_{k+n/2} ← e_k - ω^k d_k

return (y_0, y_1, y_2, ..., y_{n-1})
```

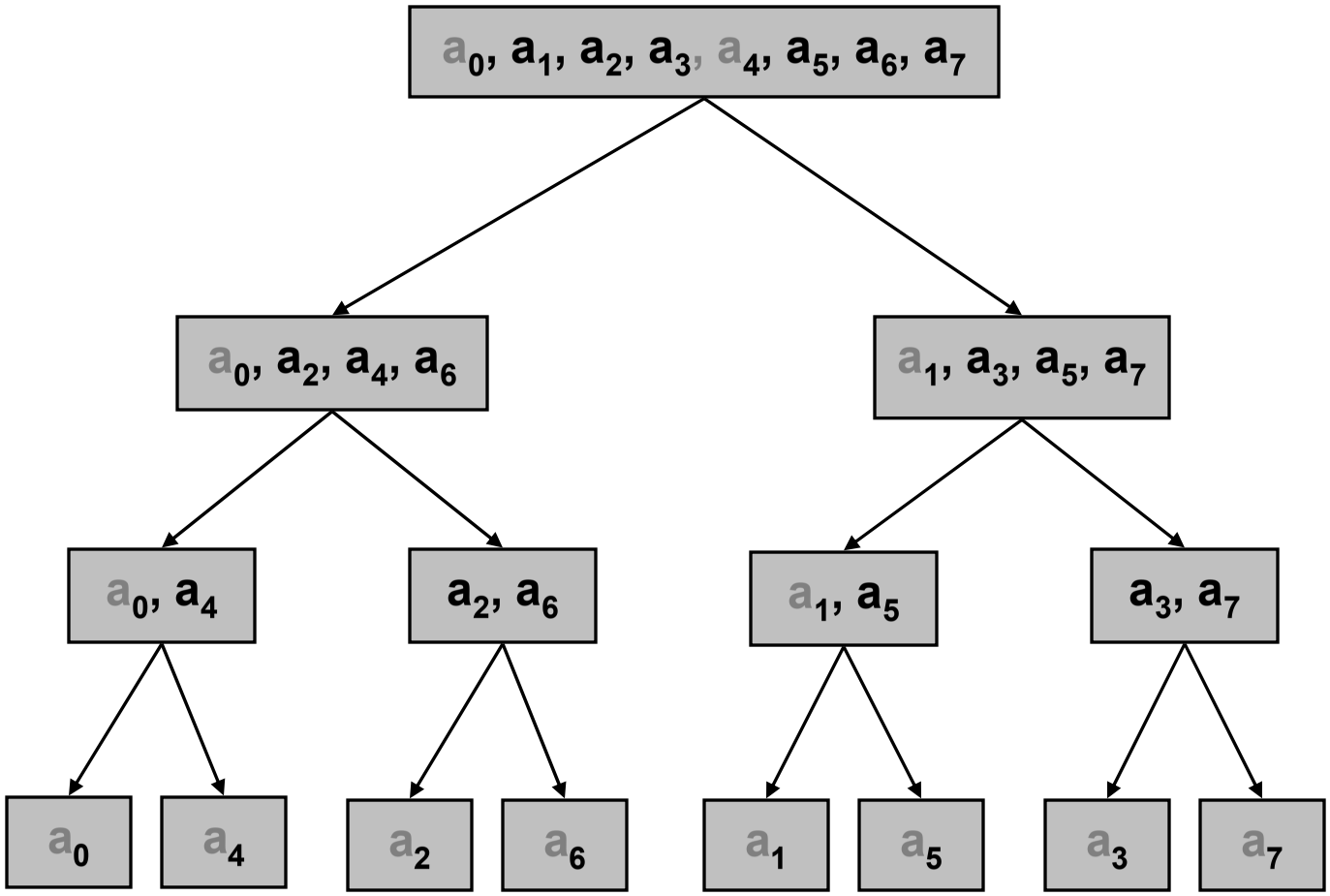
O(n) complex multiplies if we pre-compute ω^k.

$$T(n) = 2T(n/2) + O(n) \Rightarrow T(n) = O(n \log n)$$

Complexity

- Each F_k is a sum of $\log_2$ N transforms and (factors)
- There are N F_k s
- So the algorithm is $O(N \log_2 N)$
- *This is a recursive algorithm*

Scrambled Output of the FFT



"bit-reversed" order

FFTtx

```
function y = ffttx(x)
%FFTTX  Textbook Fast Finite Fourier Transform.
%   FFTTX(X) computes the same finite Fourier transform as FFT(X).
%   The code uses a recursive divide and conquer algorithm for
%   even order and matrix-vector multiplication for odd order.
%   If length(X) is m*p where m is odd and p is a power of 2, the
%   computational complexity of this approach is O(m^2)*O(p*log2(p)).

x = x(:);
n = length(x);
omega = exp(-2*pi*i/n);
if rem(n,2) == 0
   % Recursive divide and conquer
   k = (0:n/2-1)';
   w = omega .^ k;
   u = ffttx(x(1:2:n-1));
   v = w.*ffttx(x(2:2:n));
   y = [u+v; u-v];
else
   % The Fourier matrix.
   j = 0:n-1;
   k = j';
   F = omega .^ (k*j);
   y = F*x;
end
```