Plane-wave decomposition analysis for spherical microphone arrays

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Spherical Arrays

• Can provide spatial information on sound at a point
• Sound scattering by the sphere can be “undone”
• Incoming sound deduced from sound at surface
• Can be used in applications such as sound field analysis, beamforming, tracking, meeting capture and virtual reality
• Sound satisfies wave equation subject to boundary conditions

\[
\frac{1}{c^2} \frac{\partial^2 p'(r,t)}{\partial t^2} = \nabla^2 p'(r,t),
\]

\[
\nabla^2 \psi (r) + k^2 \psi (r) = 0, \quad k = \frac{\omega}{c},
\]
Sound field at a point

• Decompose sound as $\psi = \psi_{\text{in}} + \psi_{\text{scat}}$

• Incident sound field $\psi_{\text{in}}$ at sphere centre is regular

$$\psi_{\text{in}}(k; r) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_n^m R_n^m(k; r), \quad R_n^m(k; r) = j_n(kr)Y_n^m(\theta, \varphi)$$

$$Y_n^m(\theta, \varphi) = (-1)^m \sqrt{\frac{2n + 1}{4\pi} \frac{(n - |m|)!}{(n + |m|)!}} P_n^{|m|}(\cos \theta)e^{im\varphi},$$

$$P_n(s_l \cdot s_j) = \frac{4\pi}{2n + 1} \sum_{m=-n}^{n} Y_{n}^{-m}(s_l) Y_{n}^{m}(s_j).$$

• Expressions are usually truncated at $p$ terms (causing an error)
• Truncated coefficients determined from measurements
• Expansions converge and an error bound can be established
• However, expansions in spherical functions are not very informative
• Another basis can be used to explicitly express direction dependence.
Regular spherical wave functions
Plane-wave basis

• The incoming sound at a point can be expressed as a sum of plane-waves coming from all directions
  • (actually integral over all directions (unit sphere)
  \[
  \psi_{in} (r) = \frac{1}{4\pi} \int_{S_u} e^{iks \cdot r} \mu_{in} (s) dS (s),
  \]

• These functions constitute a basis as well.

• Strengths of plane-waves, \( \mu_{in} \), are known as "Herglotz wave functions" or "Signature functions"

• The two sets of basis functions can be related

\[
e^{iks \cdot r} = 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{n} i^n Y_n^{-m} (s) R_n^m (r),
\]

\[
R_n^m (r) = \frac{i^{-n}}{4\pi} \int_{S_u} e^{iks \cdot r} Y_n^m (s) dS (s),
\]
• The approximation made when an incident plane-wave is expressed in terms of spherical wave functions is

\[
\epsilon_p (s, r) = e^{iks \cdot r} - 4\pi \sum_{n=0}^{p-1} \sum_{m=-n}^{n} i^n Y_{n}^{-m} (s) R_n^m (r)
\]

\[
= \sum_{n=p}^{\infty} (2n + 1) i^n j_n (kr) P_n \left( \frac{r \cdot s}{r} \right)
\]

• Quantity can be bounded as
\[
|\epsilon_p (s, r)| \lesssim \exp \left\{ -\frac{1}{3} \left[ 2 \frac{p - kR}{(kR)^{1/3}} \right]^{3/2} \right\} = \delta_p, \quad kR \gg 1.
\]

So the difference between the "band-limited" plane-wave and full wave is

\[
|\psi_{in} (r) - \psi^{(p)}_{in} (r)| \leq \frac{1}{4\pi} \int_{S_u} |\epsilon_p (s, r)| |\mu_{in} (s)| dS (s)
\]

\[
\leq \max |\epsilon_p (s, r)| \max |\mu_{in} (s)| \lesssim \delta_p \max |\mu_{in} (s)| = \epsilon_s,
\]

For a given allowable error, this can be solved for \( p \)

\[
p \approx kR + \frac{1}{2} \left( 3 \ln \frac{\max |\mu_{in} (s)|}{\epsilon_s} \right)^{2/3} (kR)^{1/3}, \quad kR \gg 1.
\]

In multifrequency analysis, increase \( p \) along with the frequency
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FAST MULTIPole METHODS FOR THE HELMHOLTZ EQUATION IN THREE DIMENSIONS

A Volume in the Elsevier Series in Electromagnetism
• Provides relation between the plane-wave and order of spherical wave expansion needed to represent it
• Depends on frequency, size of domain and weakly on wave magnitude
• $p$ truncated plane waves used as beam patterns.
Approximate Integration (Quadrature)

- Integral over sphere must be performed discretely for truncated plane-waves
- Done via a “quadrature rule” that provides weights and points (microphone locations) that integrate spherical harmonics up to a particular order

\[ \int_{S_u} F(s) \, dS = \sum_{j=0}^{L_Q-1} F(s_j) \, w_j, \quad F(s) = \sum_{n=0}^{p-1} \sum_{m=-n}^{n} C_n^m Y_n^m(s), \]

- Gaussian quadrature of order \( p \) needs \( 2p^2 \) inconveniently distributed points
- For an arbitrary distribution we need \( 4p^2 \) points, while for some special designs we can achieve exact integration for band-limited functions
• Since both the plane-waves and the surface function $\mu_{in}$, can be expanded in spherical harmonics of order $p$, we need formula of order $2p$.
• So we need $4p^2$ or $16p^2$ points leading to a large number of microphones.
• Number $L$ for $4p^2$ shown below for different freqs. for 3cm and 9cm rad. array

<table>
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<th>$a$</th>
<th>$f$</th>
<th>$ka$</th>
<th>$p$</th>
<th>$L$</th>
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<td>4kHZ</td>
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• As exact formulae lead to large numbers of points we seek approximate ones
• Spherical designs proposed by Hardin and Sloane achieve numerically exact formulae, that however are not robust to out of band errors
• We have adopted the Fliege-Maier points/weights
• Obtained via an optimization procedure that minimize error for out-of-band functions
Solving for the Plane Wave coefficients

- Use scattering of a plane wave off a sphere
- Decompose sound-field in terms of a set of $p$ truncated plane-waves.

$$\psi_S (s; s') = K (s; s') = \frac{i}{(ka)^2} \sum_{n=0}^{p-1} \frac{i^n (2n + 1) P_n (s \cdot s')}{h'_n (ka)}.$$

Substituting in the plane-wave representation

$$\psi_S (s) = \frac{1}{4\pi} \int_{S_u} K (s; s') \mu_{in} (s') dS' = \sum_{l=0}^{L_Q-1} w_l K (s; s'_l) \mu_{in} (s'_l).$$

$$\psi_S (s_j) = \sum_{l=0}^{L_Q-1} K (s_j; s_l) w_l \mu_{in} (s_l), \quad j = 1, \cdots, L_M,$$

$L_M$ is the total number of microphones. Can be solved for $\mu_{in}$ as
Results

Here we show the reproduction of plane-waves from a particular direction in terms of other plane waves coming from a set of chosen directions.

Error is small as expected in region of spherical array.

In Duraiswami et al. (AES, 2005) we used this idea to develop HRTF-based binaural playback of sound recorded by a spherical array.

\[
\mu_{in}(s_i) = \sum_{j=0}^{L_M-1} w_j M(s_i; s_j) \psi_S(s_j),
\]

\[
M(s_i; s_j) = \frac{-i}{4\pi} \frac{(ka)^2}{p-1} \sum_{n=0}^{p-1} (2n + 1) i^{-n} h_n'(ka) P_n(s_i \cdot s_j).
\]