MODELING OF PARTICLE MOTION IN VISCOUS SWIRL FLOW BETWEEN TWO POROUS CYLINDERS

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ABSTRACT
The present paper provides a mathematical model for particle motion in viscous flow between rotating cylinders. Using an assumption that the mass fraction of particles in the flow is small, so the perturbations of the mean liquid flow due to the presence of particles is negligible, 3D liquid flow was simulated using an exact solution of the Navier-Stokes equations. Particle motion under action of drag, added mass, and buoyancy forces in the computed liquid velocity field was simulated and analyzed.

Keywords: particles, hydrodynamic forces, viscous flow, Navier-Stokes equations, titration, mathematical modeling, numerical simulations, pseudo-spectral methods.

NOMENCLATURE

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INTRODUCTION
Swirl particulate flows can be found in nature and have significant industrial applications, including in filtration and particle separation. One of the original developments of DYNAFLOW, INC. is a patented filtration system, DYNAPERM® (Chahine, 1996), that combines an annular swirl flow with filtration through microporous tubes made of thermoplastic. The present study was inspired by the need to model particle motions in such filtration systems, with the goal of developing tools for study, design, and improvement of the filtration process. The flow of liquid in the mentioned filtration system is substantially three dimensional and can be realized at a wide range of Reynolds numbers (from tens to thousands). Computation of such a velocity field is very challenging, being further complicated by its...
multiphase character and the possibility of laminar and
turbulent regimes. One of the approaches to model the
flow is based on solution of full Navier-Stokes equations.

However, the most interesting hydrodynamic prob-
lem for particle separation is prediction and control of
particle trajectories in the flow. These can substantially
deviate from the streamlines of the single-phase liquid
flow. The character of particle motion depends on char-
acteristic flow and particle relaxation times. These are
determined by the geometry and sizes of the chamber
and particles, gravity, particle density, liquid visc-
osity, etc. The problem is especially complex in case of
higher particle concentration. For lower concentrations
we may neglect the influence of particles on the carrier
liquid flow.

In the present study we found an exact solution of
the Navier-Stokes equations applicable for the case of
swirl flow between annular porous pipes under a pres-
sure gradient. This solution, generally, is not analyti-
cal and can be computed by solving a boundary value
problem for a nonlinear ordinary differential equation.
A similar solution, but in some simpler cases was ob-
tained and analyzed by Gold'shtik & Ersh (1991). The
found solution reduces to analytical solutions in some
limiting cases.

Exact solutions of the Navier-Stokes equations are
valuable since they can be used for stability studies
(Gold'shtik & Ersh, 1991, 1992) and as a non-trivial
three-dimensional solution for testing of 3D Navier-
Stokes solvers (Ethier & Steinman, 1994). To solve the
obtained system of ODE we developed a method based
on pseudo-spectral technique which has an advantage in
comparison with shooting methods used by Gold'shtik
& Ersh (1991), because the minimization with respect
to multiple shooting parameters may be ill-posed.

FORMULATION OF THE PROBLEM

Flow Geometry

The geometry of the flow is shown in Figure 1. Two
vertical porous coaxial cylinders of radii \( R_i \) and \( R_o \),
rotate with prescribed velocities \( V_{ti} \) and \( V_{to} \). The
liquid-particle mixture flow into and out of the domain
through the annuli at the bottom and top and through
the cylinder walls with the prescribed radial velocities
\( V_{ri} \) and \( V_{ro} \).

Basic Assumptions

To model a flow of liquid with suspended solid parti-
cles we accept the following basic assumptions:

1. The liquid is viscous and incompressible and its
   motion can be described by the Navier-Stokes
equations;

2. The particles are rigid and spherical;

3. The mass and volume fraction of particles in the
   mixture is small enough to neglect the influence of
   particles on the liquid motion;

4. The velocity field of liquid is steady and the ve-
ocity components do not depend on the angular
variable;

5. Several forces acting on the particles, such as the
   Dasset force, Magnus-Joukowski force and side-
wise force are negligibly small comparing to viscous
   drag, inertial, and gravitation forces.

Liquid Motion

Using the basic assumptions we can describe the flow
in the cylindric coordinates \((r, z, \theta)\) by the following
equations

\[
\begin{align*}
V_r \frac{\partial V_r}{\partial r} - \frac{V_r^2}{r} + V_r \frac{\partial V_r}{\partial z} &= -\frac{1}{\mu} \frac{\partial p}{\partial r} + \nu_1 \mathcal{L}_1 V_r, \\
V_r \frac{\partial V_\theta}{\partial r} + \frac{V_r V_\theta}{r} + V_z \frac{\partial V_\theta}{\partial z} &= \nu_1 \mathcal{L}_1 V_\theta, \\
V_r \frac{\partial V_z}{\partial r} + V_z \frac{\partial V_z}{\partial z} &= \frac{1}{\rho_i} \frac{\partial p}{\partial z} + \nu_1 \mathcal{L}_2 V_z - g,
\end{align*}
\]

Figure 1: Problem definition sketch.
where

\[ L_1 V = \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} \frac{\partial V}{\partial r} \right) + \frac{\partial^2 V}{\partial z^2} \right], \]

\[ L_2 V = \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial V}{\partial r} \right) + \frac{\partial^2 V}{\partial z^2} \right], \]

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V}{\partial r} \right) + \frac{\partial^2 V}{\partial z^2} = 0, \]

which are subject to the following boundary conditions at the walls of cylinders,

at \( r = R \):

\[ V_r = V_{ri}, \quad V_\theta = V_{ti}, \quad V_z = 0, \quad (2) \]

at \( r = R_o \):

\[ V_r = V_{ro}, \quad V_\theta = V_{to}, \quad V_z = 0, \]

and conditions at the entrance and exit of the annular space, which we specify in every particular case of exact solution. Here \( \mathbf{V}_l = (V_r, V_\theta, V_z) \) is the liquid velocity, \( p \) is the pressure, \( \rho_l \) and \( \nu \) are the density and kinematic viscosity, and \( g \) is the gravity acceleration.

**Particle Motion**

Due to small particle concentration in the mixture the motion of the particulate phase motion can be described as that of a set of individual particles which do not interact with each other. In this case the motion of a single spherical particle of radius \( a \) in a known liquid velocity field can be described by

\[ \frac{4}{3} \pi a^3 \rho_p \frac{d\mathbf{V}_p}{dt} = \mathbf{F}, \quad \frac{d\rho}{dt} = \frac{\partial}{\partial t} + \mathbf{V}_p \cdot \nabla, \quad (3) \]

where \( m_p, \rho_p \) and \( \mathbf{V}_p \) are the particle mass, density and velocity, and \( \mathbf{F} \) is the total force acting on the particle, and \( t \) the time. The full derivative \( d\mathbf{V}_p/dt \) along the particle trajectory means the full acceleration and, particularly, includes the centrifugal acceleration. For steady particle motion \( \frac{d\mathbf{V}_p}{dt} = \mathbf{V}_p \cdot \nabla \) and this can be written in the cylindric coordinates in the form similar to the right hand sides of equations for liquid motion (1).

According to the basic assumptions we represent the force as

\[ \mathbf{F} = \mathbf{F}_o + \mathbf{F}_A + \mathbf{F}_m + \mathbf{F}_D, \quad (4) \]

with the particle weight given by

\[ \mathbf{F}_g = \frac{4}{3} \pi a^3 \rho_p g, \quad (5) \]

and the force connected with the far field liquid acceleration

\[ \mathbf{F}_A = \frac{4}{3} \pi a^3 \rho_l \left( \frac{d\mathbf{V}_l}{dt} - g \right), \quad (6) \]

\[ \frac{d\mathbf{V}_l}{dt} = \frac{\partial}{\partial t} + \mathbf{V}_l \cdot \nabla. \]

For a steady flow of liquid we have \( \frac{d\mathbf{V}_l}{dt} = \mathbf{V}_l \cdot \nabla \). Note that the liquid acceleration also can be expressed from the Navier-Stokes equations in the form

\[ \frac{d\mathbf{V}_l}{dt} = -\frac{1}{\rho_l} \nabla p + \nu \nabla^2 \mathbf{V}_l + \mathbf{g}. \quad (7) \]

However the right hand side of this equation requires simulation of the pressure field and higher derivatives of the velocity and is less convenient from the point of view of numerical methods than simulation of \( (\mathbf{V}_l \cdot \nabla) \mathbf{V}_l \). It also follows from (5), (6), and (7) that for the liquid at rest \( (\mathbf{V}_l = 0) \) we have the hydrostatic pressure profile, \( \nabla p = \rho \mathbf{g} \), and \( \mathbf{F}_A = -\frac{4}{3} \pi \rho_a^3 \rho_p \mathbf{g} \) represents the static buoyancy force.

\[ \mathbf{F}_m = \frac{2}{3} \pi a^3 \rho_l \left( \frac{d\mathbf{V}_l}{dt} - \frac{d\rho}{dt} \right) \]

is the virtual mass force; and

\[ \mathbf{F}_D = \frac{1}{2} \pi a^2 \rho_l C_D (Re) (\mathbf{V}_l - \mathbf{V}_p) |\mathbf{V}_l - \mathbf{V}_p| \]

is the steady drag force.

The Basset force, \( \mathbf{F}_B \), plays a significant role in attenuation of small amplitude high-frequency perturbations in particulate systems (Gumerov et al, 1988). However in the vast majority cases it can be neglected. Indeed, for small Reynolds numbers of particle motion we have shown (Gumerov et al, 1988; Nigmatulin, 1991)

\[ \frac{|\mathbf{F}_B|}{|\mathbf{F}_m|} < \left| \frac{|\mathbf{F}_D|}{|\mathbf{F}_B|} \right| \sim \frac{a}{\sqrt{\nu \tau_*}} = K_v, \]

where \( \tau_* \) is the characteristic time of particle motion, and \( |\mathbf{F}_B| < |\mathbf{F}_D + \mathbf{F}_m| \) in both limits of small and high values of parameter \( K_v \). This means that both for slow flows and flows with high acceleration the sum \( \mathbf{F}_D + \mathbf{F}_m \) makes the major contribution to the total force connected with the particle relative motion. For higher Reynolds numbers the influence of the Basset force is very small, because it acts only during the time of development of the unsteady viscous boundary layer of order \( \tau_* \sim a^2 / (\nu \tau_*) \) that corresponds to \( K_v \sim \sqrt{Re} \gg 1 \).

Various approximations of the standard drag curve for rigid spherical particles can be found in book of Clift et al (1978). In the present study we used the following approximation of the drag coefficient \( C_D \) valid
in a wide range of Reynolds numbers, Re, of particle motion \( \lesssim 3 \cdot 10^5 \):

\[
C_D(Re) = \frac{24}{Re} + \frac{4}{\sqrt{Re}} + 0.4, \quad \text{Re} = \frac{2a |V_i - V_p|}{\nu}.
\]

Using the above expressions we can represent equation of particle motion in the form

\[
\frac{dpV_p}{dt} = \frac{2(\rho_p - \rho_\ell)}{2\rho_p + \rho_\ell} \cdot \frac{3\mu_i}{2\rho_p + \rho_\ell} \frac{d^2 V_i}{dt^2} + \frac{3\mu_i C_D(Re)}{4(2\rho_p + \rho_\ell) a} (V_i - V_p) |V_i - V_p|.
\]

**DIMENSIONLESS PARAMETERS**

To characterize various possible liquid flow regimes we choose the following dimensionless quantities

\[
\eta = \frac{r}{R_o}, \quad \zeta = \frac{z}{R_o}, \quad u_l = \frac{V_l}{V_\ell}, \quad \eta_l = \frac{R_l}{R_o},
\]

\[
\mathcal{R} = \frac{R_o V_*}{R_\ell}, \quad P = \frac{p - p_\ell}{\rho_\ell V_*^2}, \quad G = \frac{R_o g}{V_*^2}.
\]

\[
\eta_{to} = \frac{V_\omega}{V_*}, \quad \eta_{ro} = \frac{V_\omega}{V_*},
\]

\[
\eta_{ti} = \frac{V_{ti}}{V_*}, \quad \eta_{ri} = \frac{V_{ri}}{V_*},
\]

where \( p_\ell \) is the characteristic pressure, \( V_* \) is the scaling velocity, and \( \mathcal{R} \) is the Reynolds number of the flow of liquid. To characterize particle motion we introduce the following dimensionless variables and constants

\[
\tau = \frac{V_* t}{R_o}, \quad u_p = \frac{V_p}{V_*}, \quad \rho = \frac{\rho_p}{\rho_\ell}, \quad \epsilon = \frac{a}{R_o}.
\]

**SOLUTION OF NAVIER-STOKES EQUATIONS**

**Reduction to a system of ODEs**

The above equations with boundary conditions allow us to search solutions in the following form

\[
\eta_r = u_r(\eta), \quad u_z = u_z(\eta) + u_\ell(\eta) \zeta, \quad \eta_\theta = u_\theta(\eta), \quad P = \Pi(\eta) - \frac{1}{2} \alpha \epsilon^2 - \beta \zeta.
\]

Separating independent variables \( \eta \) and \( \zeta \) we have from the Navier-Stokes equation

\[
\frac{1}{\eta} (\eta u_r)' + u_z = 0, \quad (13)
\]

\[
u_\eta u_\eta' + \frac{\eta_\theta'}{\eta} + \Pi' = \mathcal{R}^{-1} \left( \frac{1}{\eta} (\eta u_r) \right)' \quad (14)
\]

\[
u_r u_r' + \frac{\eta_\theta u_\theta'}{\eta} = \mathcal{R}^{-1} \left( \frac{1}{\eta} (\eta u_r) \right)' \quad (15)
\]

\[
u_r u_z' + \frac{u_\theta}{\eta} = \mathcal{R}^{-1} \left( \frac{1}{\eta} (\eta u_r) \right)' \quad (16)
\]

\[
u_r u_\theta' + \frac{u_z}{\eta} = \mathcal{R}^{-1} \left( \frac{1}{\eta} (\eta u_r) \right)' \quad (17)
\]

where the prime denotes differentiation with respect to \( \eta \), and which are subject to at \( \eta = \eta_1 \)

\[
u_r = \nu_{r1}, \quad \eta_\theta = \eta_{t1}, \quad u_z = u_\ell = 0, \quad (18)
\]

and at \( \eta = 1 \)

\[
u_r = \nu_{ro}, \quad \eta_\theta = \eta_{to}, \quad u_z = u_\ell = 0. \quad (19)
\]

We see that equations (13) and (17) form a closed third order system in \( u_r \) and \( u_{r2} \). This subsystem is subject to four boundary conditions, but contains the unknown constant \( \alpha \). Differentiating equation (17) eliminates \( \alpha \) and makes the number of boundary conditions consistent. Using (13) we also can express \( u_{r2} \) through \( u_r \) and its derivative and obtain the following fourth order differential equation:

\[
u_r u_r'' + \frac{2}{\eta} u_r''' - \frac{3}{\eta^2} u_r'' + \frac{3}{\eta^3} u_r' + \frac{3}{\eta^4} u_r = \mathcal{R} \left[ u_z u_r' \left( \frac{1}{\eta} u_r' - rac{1}{\eta^2} u_r'' \right) - \frac{3}{\eta^3} u_r u_r' + \frac{4}{\eta^4} u_r' \right],
\]

subject to

\[
u_r |_{\eta = \eta_1} = u_{r1}, \quad u_r' |_{\eta = \eta_1} = 0, \quad u_r' |_{\eta = 1} = u_{ro}, \quad u_r' + u_r |_{\eta = 1} = 0. \quad (21)
\]

The parameter \( \mathcal{R} \) controls the strength of the nonlinearity. Once we solve equation for \( u_r \), we can evaluate \( u_{r2} \) using (13), while the constant \( \alpha \) can be evaluated from (17). To solve the remaining equations we can consider \( u_r \) and \( u_{r2} \) as given, thereby rendering them linear. Thus, we can determine \( u_z \) from (16), \( \eta_\theta \) from (15), and \( \Pi \) from (14). Note that \( \Pi \) can be found up to a constant, which can be specified from known ambient pressure.

For the case \( \beta = G \) we have

\[
u_z \equiv 0, \quad u_z |_{z=0} = 0. \quad (22)
\]

This particular case was used in stability studies of flow in a rotating porous pipe without internal cylinder in (Gold'shtik & Ersh, 1991, 1992).
Particular Analytical Solution

In case
\[ u_{r2} = 0, \quad \eta = 0, \]
the radial velocity can be found by integration of equation (13):
\[ u_r = \epsilon \eta^{-1}, \quad (23) \]
where \( \epsilon \) is a constant of integration. This case is particular and can be realized only if
\[ u_{r0} = u_{ri} \eta_i = \epsilon. \quad (24) \]
Choosing \( V_r = |V_{r0}| \) as the velocity scale
\[ R = \frac{R_o}{|V_{r0}|} = \frac{R_i}{|V_{r1}|}, \]
\[ c = \text{sgn} V_{r0} = \pm 1. \quad (25) \]
Substituting (23) and (24) to (15) and (16) we find
\[ u_{z1} = \frac{R (G - \beta)(\eta_i^2 - 1)}{2(2 - cR)} \times \]
\[ \left( \frac{\eta_i^2 - 1}{\eta_i^2 - 1} - \frac{cR - 1}{\eta_i^2 - 1} \right), \quad (26) \]
\[ u_{\theta} = \frac{1}{\eta} \left[ u_{r0} + (u_{z1} - u_{r0}) \frac{\eta_i^2 + cR - 1}{\eta_i^2 + cR - 1} \right]. \quad (27) \]

At the limit \( V_{r0} \to 0 \) (\( R \to 0 \), \( G - \beta \sim R^{-2} \)) this solution transits to the well-known result for flow between two rotating coaxial cylinders. Note that in the case \( R \to 2 \) the above solution has a limit form containing logarithms.

NUMERICAL METHODS

Chebyshev Collocation Solution of ODEs

Equations (20), (15), and (16) are solved using Chebyshev collocation (Canuto et al, 1987). Accordingly we introduce the Chebyshev Gauss-Lobatto points \([x_0, x_M]\) (which are mapped to \([-1, 1]\)). The differential equation is collocated at the points \( x_1 \) through \( x_{M-1} \), while the boundary conditions are collocated at points \( x_0, x_M \). The use of cardinal Chebyshev functions (Boyd, 1988) allows the development of matrix operators which directly act on the function values defined on the Gauss-Lobatto grid and produce the values of the appropriate derivatives on this grid. This allows direct translation of the analytic form of a differential equation into its pseudospectral form, and thus the method and algorithms possess an easy translation between the forms.

Let \( D^M \) denote the matrix Chebyshev differentiation operator associated with the \( M + 1 \) Gauss-Lobatto collocation points. The entries of this matrix are known analytically (see e.g., Boyd, 1988). This matrix operator acts on the vector of function values given at the \( M + 1 \) Gauss-Lobatto points. Also, let \( D^2 = D^M D^M \) represent the matrix operator yielding the second derivative at the collocation points. We similarly define the operators \( D^3 \) and \( D^4 \).

The above system of equations can be written by collocating the differential equation and boundary conditions at the appropriate Gauss-Lobatto points, to yield a matrix representation of the linear portion of the operators and boundary conditions. The linear matrix representation can then be \( LU \) decomposed, and used in an iterative procedure to solve the nonlinear system.

Solution of Nonlinear ODEs for Liquid Motion

To solve the nonlinear system of equations of liquid motion we can employ a simple iterative strategy. The linear system corresponding to vanishing \( R \) is solved. Using this solution as guess we evaluate the nonlinear term and solve the system treating the nonlinear term as given, and iteratively update the guess. To ensure that the procedure converges, the value of \( R \) is increased gradually to the final value following a relaxation.

Solution of ODEs for Particle Motion

The components of liquid velocity \( u_r, u_{\theta}, u_{z1}, \) and \( u_{z2} \) and their derivatives obtained by the above method are used to determine the liquid velocity and acceleration at any given point by interpolation. The equations of particle motion can be written as a system of 6 ODEs (3 coordinates and 3 velocity components). Then the system can be solved by a standard Runge-Kutta 4th order solver.

RESULTS OF SIMULATIONS

Liquid Motion

The developed solver was tested by comparison of the numerical results with the analytical solution described above. These comparisons showed a good convergence at the Reynolds numbers of radial motion, \( R \), in range 0 - 20 and the Reynolds numbers of tangential motion 0 - 1000. Since the liquid motion depends on 7 dimensionless parameters, a complete analysis of all types of flow patterns is hard to perform. Instead we will take a single example range of parameters and discuss the characteristics of particle motion for this case.

The particular example chosen corresponds to
\[ \eta_i = 0.2, \quad R = 1, \quad \beta - G = 0. \quad (29) \]
Particle Motion

The particle motion in a prescribed flow depending on 7 parameters depends on additional 9 parameters: \( \rho, \epsilon, G, 3 \) initial particle coordinates, \( x_{p0} = (x_{p0}, y_{p0}, z_{p0}) \), and 3 initial particle velocity components \( u_{p0} = (u_{x0}, u_{y0}, u_{z0}) \). In the selected for illustration cases we fixed \( \epsilon = 0.02, G = 1.2 \cdot 10^4 \), and prescribed the initial particle velocity to be equal the liquid velocity at the starting point: \( u_{p0} = u_l(x_{p0}) \). Also in all cases we used the liquid field with parameters (29) and plotted 3D pictures of particle trajectories.

The deviation of particle trajectories from a streamline can be characterized by the following parameters:

\[
K_S = \frac{\tau_S}{\tau_{\text{min}}} = \frac{2}{9} \rho \varepsilon R_{\text{max}},
\]

\[
\tau_S = \frac{2 \rho \varepsilon^2}{9 \mu u_l}, \quad \tau_{\text{min}} = \frac{R_{\text{e}}}{V_{\text{max}}},
\]

\[
R_{\text{max}} = \frac{R_{\text{e}} V_{\text{max}}}{u_l},
\]

where \( \tau_S \) is the Stokes velocity relaxation time, \( \tau_{\text{min}} \) is the minimal characteristic time of the flow, associated with the maximal velocity, \( V_{\text{max}} \), and \( R_{\text{max}} \) is the Reynolds number based on \( V_{\text{max}} \). In our simulations this parameter was small, \( K_S \lesssim 10^{-2} \). However, we found that the particle trajectories essentially deviates from the trajectories of liquid particles. This can be explained by substantial accelerations realized in the calculated case. For instance, the dimensionless parameter characterizing gravitation, \( G \), was large in all cases.

Figures 4-6 demonstrate the dependence of particle trajectory on the starting point, \( x_{p0} \), for \( \rho = 2.5 \). In all cases the starting point was selected to be on the surface of the outer cylinder. The difference is only in \( z_{p0} \) which was equal to 0.8, 0.9, and 1 respectively Figures 4, 5 and 6 correspondinglly. At \( z_{p0} = 0.8 \) the particle sinks (gravitation forces in the axial direction dominate the hydrodynamic forces). At \( z_{p0} = 0.9 \) the particle experiences the following curious motion in \( z \)-direction: it goes down, then up, and then down again. In this case there exists an approximate balance between projections of gravitation and hydrodynamic forces in \( z \)-direction. At \( z_{p0} = 1 \) the same qualitative behavior takes place but with hydrodynamic forces dominating.

The cases of non-monotonic motion of particles in \( z \)-direction can be understood from the character of the \( u_{z2} \) velocity profile shown in Figure 2. The forces acting upward are much smaller near the cylinder walls than in the middle of the annulus, while the gravitational forces acting in the opposite direction are constant. This results in a change of sign of the net force acting in the vertical direction.

Figure 2: Dimensionless velocity components for the example case considered.

\[
u_{x0} = 30, \quad u_{y0} = -1, \quad u_{z0} = 0, \quad u_{z1} = -1,
\]

which has zero axial velocity at the bottom of the tube, \( z = 0 \) (22). The values of the velocity components are shown in Figure 2. Figure 3 shows a 3D picture of a streamline corresponding to the calculated velocity field. This streamline starts from the outer blowing porous cylinder at \( z = 1 \) and ends at the inner sucking porous cylinder at \( z = 5 \). Note that in case (22) the motion is self-similar with respect to deformation of \( z \). In other words, to obtain a streamline starting, say at \( z = 0.2 \) and ending at \( z = 1 \), we just need to compress by five times the streamline shown in Figure 3. Generally, if \( \beta \neq G \) the flow is not self-similar.

The obtained solution shows that the liquid accelerates in the axial direction. This always occurs if \( \eta \mu \tau_{\text{rel}} \neq u_{\tau_0} \), since in such cases the liquid influx from the blowing cylinder is not equal to the flux through the sucking cylinder. The difference in these fluxes adds (or subtracts) some liquid in each section, that can then flow only in the axial direction. Since the liquid is incompressible this can be realized only through the increasing (or decreasing) of the flow rate in the axial direction.
Figures 7 and 8 show simulations for heavy particles with $\rho = 10$. In this case the direction of radial motion of the particle depends on the starting point. In Figure 7 the starting point is $x_{f0} = (0.4,0,3.7)$ and the heavy particle moves in the direction from the inner to the outer cylinder. It is consistent with the above comment on the gravitation forces, that it first goes up and then down near the outer wall. In Figure 8 the starting point is located slightly closer to the inner wall, $x_{f0} = (0.35,0,5)$, and the particle moves inward. Again the particle motion in z-direction is non-monotonic: the particle rises and then goes down near the cylinder wall.

**CONCLUSIONS**

- The results obtained show that even in the simplified model used in the present study the behavior of liquid and particles is controlled by many parameters (7 for liquid plus 9 for particle). Depending on these parameters a particle can experience various types of motion including motion with changing direction. Such behavior of particles is typical for cyclones and centrifuges and can be explained by the obtained velocity profiles of the liquid.

- The exact and analytical solutions of the Navier-Stokes equations obtained can be used for modeling of swirl flows between porous coaxial cylinders. This can be used in various industrial devices using swirl flows for filtration, particle separation, etc. Applications are also possible for atmospheric and marine flows.

- The modeling approach and the iterative pseudo-spectral numerical procedures developed for solution of non-linear boundary value problems showed good potential for simulation and prediction of particle motion. They can be recommended for studies of particle motion in viscous flows with other geometry and configurations.

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**References**


Figure 3: A streamline of the liquid motion for the example case considered passing through $x_{p0} = (1, 0, 1)$. The streamline is shown as the dark line, the inner cylinder is dark gray, while the space between the cylinders is light gray.

Figure 4: Trajectory of a particle with $\rho = 2.5$ and $x_{p0} = (1, 0, 0.8)$.

Figure 5: Trajectory of a particle with $\rho = 2.5$ and $x_{p0} = (1, 0, 0.9)$.

Figure 6: Trajectory of a particle with $\rho = 2.5$ and $x_{p0} = (1, 0, 1)$.

Figure 7: Trajectory of a particle with $\rho = 10$ and $x_{p0} = (0.4, 0, 3.7)$.

Figure 8: Trajectory of a particle with $\rho = 10$ and $x_{p0} = (0.35, 0, 5)$.