Finite-Difference and Fast Multipole Methods for HRTF

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Outline

• Motivation
• Finite-Difference Method and The Perfectly Matched Layer (PML)
• Multidimensional Applications
• Results: A Simplified, Spherical Head Model
• Future Work:
  – FD simulation of a realistic head model
  – Fast Multipole Method for HRTF
• Summary
Motivation

- **Objective:** To calculate the acoustic wave field excited by any sources in an inhomogeneous medium (such as a head).

- **Features:**
  - Arbitrary spatial distribution
  - Arbitrary temporal excitation
  - Wide-band calculation
  - High resolution model
  - High Accuracy Requirement

- **Approaches**
  - Finite-difference (PDE) method
  - Fast multipole (Integral Equation) method
Finite-Difference Method

- Acoustic Wave Equations for Absorptive Media

\[ \rho \frac{\partial \mathbf{v}(\mathbf{r}, t)}{\partial t} = -\nabla p, \quad (1) \]

\[ \frac{\partial p}{\partial t} + \gamma(\mathbf{r})c^2(\mathbf{r})p(\mathbf{r}, t) = -\rho(\mathbf{r})c^2(\mathbf{r})\nabla \cdot \mathbf{v}(\mathbf{r}, t) + f_s(\mathbf{r}, t), \quad (2) \]

- Finite Difference Approximation

\[ \frac{\partial p(x)}{\partial x} \approx \frac{p(x + \Delta x/2) - p(x - \Delta x/2)}{\Delta x} \]

- Similar approximations for other derivatives.
• A staggered finite-difference grid
Absorbing Boundary Condition

- Perfectly matched layer first proposed by Berenger (1993) for electromagnetic waves.

- Chew and Liu proved that PML also applies to elastic waves, in spite of P and S coupling (1995).
  - Hastings, Schneider and Broschat also implemented 2D PML ABC, but using potentials (1996).

- PML ABC is extended to acoustic waves in absorbptive media Liu (1997).

- HRTF is an application of this more general work by Liu (1997)
  
  Finite difference approximation is used for derivatives
A Perfectly Matched Interface

- A half-space with loss.
- Waves attenuate in the direction normal to the interface.
- No reflection for all frequencies and all incidence angles, including evanescent waves.
Stretched Complex Coordinates

- Let \( z = z'e_z \) where \( e_z = (a_z + i\frac{\omega z}{\omega}) \) is the “stretching constant”
- Then \( e^{ikz} \) becomes \( e^{ika_z z' - k z' \omega_z / \omega} \).
- A wave become attenuated in the \( z' \) coordinate, if \( e_z \) is complex.
- A PML interface:
• Reflection coefficient at the interface is

\[ R = \frac{\left( \frac{k_x}{e_{2x}} \right) \left( \frac{k_{1z}}{e_{1z}} \right) - \left( \frac{k_x}{e_{1x}} \right) \left( \frac{k_{2z}}{e_{2z}} \right)}{\left( \frac{k_x}{e_{2x}} \right) \left( \frac{k_{1z}}{e_{1z}} \right) + \left( \frac{k_x}{e_{1x}} \right) \left( \frac{k_{2z}}{e_{2z}} \right)}, \quad \text{where} \quad k_i^2 = \left( \frac{k_x}{e_{ix}} \right)^2 + \left( \frac{k_{iz}}{e_{iz}} \right)^2 \]

If we choose \( k_1 = k_2, e_{1x} = e_{2x}, \) but \( e_{1z} \neq e_{2z}, \) then \( R = 0. \)

• Therefore, region 1 with \( e_{1x} = e_{1z} = 1 \) is regular, but region 2 with \( e_{2x} = 1, e_{2z} = \text{complex} \) is a PML region. **Zero reflection!**

• Such a layer is called the perfectly matched layer (PML).
• In stretched complex coordinates, we replace $\nabla$ by

$$\nabla_e = \sum_{\eta=x,y,z} \hat{\eta} \frac{1}{e_\eta} \frac{\partial}{\partial \eta}, \quad e_\eta = a_\eta + i \frac{\omega_\eta}{\omega}$$

• Split acoustic pressure field: $p = \sum_{\eta=x,y,z} p^{(\eta)}$

• Split acoustic wave equations

$$-i\omega \rho \mathbf{v}(\mathbf{r}, \omega) = -\sum_{\eta=x,y,z} \hat{\eta} \frac{1}{e_\eta} \frac{\partial p(\mathbf{r}, \omega)}{\partial \eta}$$

$$-i\omega p(\mathbf{r}, \omega) + \gamma(\mathbf{r}) c^2(\mathbf{r}) p(\mathbf{r}, \omega)$$

$$= -\rho(\mathbf{r}) c^2(\mathbf{r}) \sum_{\eta=x,y,z} \frac{1}{e_\eta} \frac{\partial v_\eta(\mathbf{r}, \omega)}{\partial \eta} + f_s(\mathbf{r}, \omega)$$
• Split Equations in Time Domain

\[ a_\eta \rho \frac{\partial v_\eta}{\partial t} + \omega_\eta \rho v_\eta = -\frac{\partial p}{\partial \eta} \]

\[ a_\eta \frac{\partial p^{(\eta)}}{\partial t} + (a_\eta \gamma c^2 + \omega_\eta)p^{(\eta)} + \omega_\eta \gamma c^2 \int_{-\infty}^{t} p^{(\eta)}(\mathbf{r}, t') dt' \]

\[ = -\rho c^2 \frac{\partial v_\eta}{\partial \eta} + f_s^{(\eta)}(\mathbf{r}, t) \]

• **FDTD Method:** Using FD to Approximate These Split Equations
Performance of PML ABC
(a) Relative Maximum Amplitude (dB) versus Time Step

(b) Normalized Absolute Pressure (dB)

(c) Normalized Absolute Pressure (dB) for different n values

(d) Normalized Absolute Pressure (dB) for different n values
A 2-D Cylinder (0.4 m)
(a) Normalized Pressure vs. Time (s)
(b) Normalized Pressure vs. Time (s)

- - - Analytical

- - - Numerical

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A Large 2-D Shallow Water Simulation
(23.36 million unknowns)
Results: A Simplified, Spherical Head Model

- As a first step, model the head as a spherical body

![Diagram showing a spherical head model with source and receivers coordinates.](image)
A. A Rigid Spherical Head

- Excellent agreement with analytical solution.
B. A Penetrable Spherical Head (More Realistic)

- Excellent agreement with analytical solution.
Future Work

- A Realistic Head Model
- Fast Multipole Method (FMM) For Integral Equation

\[ p(r) H(r - r_s) = p^{\text{inc}}(r) + \int_{s} ds' \left[ g(r - r') \frac{\partial p(r')}{\partial n'} - \frac{\partial g(r - r')}{\partial n'} p(r') \right] \]
Summary

- A highly accurate finite-difference time-domain method has been developed with PML as the absorbing boundary condition.
- Excellent agreement has been achieved in benchmark examples.
- Acoustic wave interaction with a realistic head model can be modeled up to 6–8 kHz. This will be done in near future.
- Future work: FMM solution for HRTF.