1 Problem (Homework 2)

Let

$$\Phi_{ji} = \mathrm{sinc}\,(y_j - x_i)\,, \quad i = 1, ..., N, \quad j = 1, ..., M,$$
$$\mathrm{sinc}\,(x) = \frac{\sin x}{x}, \quad \mathrm{sinc}\,(0) = 1,$$

$$\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} & ... & \Phi_{1N} \\ \Phi_{21} & \Phi_{22} & ... & \Phi_{2N} \\ ... & ... & ... & ... \\ \Phi_{M1} & \Phi_{M2} & ... & \Phi_{MN} \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ ... \\ u_N \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ ... \\ v_M \end{pmatrix}, \tag{1}$$

$x_1, ..., x_N$, $y_1, ..., y_M, u_1, ..., u_N$, are random numbers distributed uniformly in $[0, 1]$. Compute the matrix-vector product

$$\mathbf{v} = \Phi\mathbf{u}, \tag{2}$$

or

$$v_j = \sum_{i=1}^{N} \Phi_{ji} u_i, \quad j = 1, ..., M, \tag{3}$$

with absolute error $\epsilon < 10^{-6}$. The matrix sizes, $N, M > 0$ are given (fixed) positive integers.

1. Write down a factored expression. Estimate the error in truncating the series using residual term evaluation for the Taylor series, and evaluate the truncation number, p,as a function of the required accuracy and N. Provide a formula that can be used for the "fast" (Middleman) ($O(N + M)$) method.

2. Write a program that implements both the "brute-force" computation based on Eq. (3) and the "fast" method.

3. Plot the absolute maximum error between the brute force and Middleman methods for $N = 10^3$ and $M = 2N$ and p varying between 1 and 11. Compare the results with your evaluations of the accuracy.

4. Take equispaced distribution of points, $x_i = \frac{i-1}{N}, \quad i = 1, ..., N, \quad y_j = \frac{j-1}{2N}, \quad j = 1, ..., 2N$. In Matlab Help find function "conv", learn it and implement an alternative method of computing matrix-vector product for the given matrix. This function provides fast $O(N \log N)$ convolution for equispaced data based on the Fast Fourier Transform. You do not have to learn the FFT implementation, but you need to learn how to use this fast convolution. Note that the Matlab fft function is highly optimized and the result can be really fast, so you can go to very large (million size N). We do not need to do this but we would like to comapre asymptotic complexities of the three methods, brute-force, Middleman, and FFT-convolution.

5. Provide a graph that compares the CPU time required by the brute-force, Middleman, and FFT-convolution methods. Plot all comparisons in "log-log" scales. It is enough to provide just a few points for N varying between 10^2 and 10^3 for the brute-force method to get the asymptotic $O(N^2)$ behavior, which can be extrapolated to the larger N. Compute using Middleman for several $N's$ varying between 10^2 and 10^4 and compute the FFT-convolution for $N's$ varying between 10^2 and 10^5 or until you see approximately linear scaling ($O(N \log N)$ behaves almost as a linear function). For the Middleman computations use the theoretical value of the truncation number that ensures that the required accuracy is achieved.

6. Provide a graph of the abs. max. error (between the brute force and two fast methods) for N varying between 10^2 and 10^3, $M = 2N$ and the truncation numbers used for each N.

Hints.

1. Note that each source contributes to the error. So the truncation number p,corresponding to a required accuracy ϵ, depends on N . This relationship is an implicit function of p and you can either solve for p (write a Matlab function to do that) or determine it by developing a table of values and interpolating.
2. You may find that to use Mtlab function “conv” you need to use an extended size of arrays padded with zeros.
3. The maximum absolute error is defined as

$$error = \max_{i=1,\dots,N} \left| v_i^{brute-force} - v_i^{fast} \right|. \tag{4}$$

   Plot the theoretical error bound on the same graph (use hint 1).
4. You may keep the truncation number constant (using the one evaluated for $N \leqslant 10^4$) or vary it with N according to the theoretical estimate for the error. In this case the function calculated in hint 1 will be helpful.
5. To minimize the truncation number for the Middleman method, think where should be the center of expansion located.