Compute the matrix-vector product

$$\mathbf{v} = \mathbf{\Phi u}, \qquad v_j = \sum_{i=1}^{N} \Phi_{ji} u_i, \quad j = 1, ..., M, \tag{1}$$

with absolute error $\epsilon < 10^{-6}$, where

$$\mathbf{\Phi} = \begin{pmatrix} \Phi_{11} & \Phi_{12} & ... & \Phi_{1N} \\ \Phi_{21} & \Phi_{22} & ... & \Phi_{2N} \\ ... & ... & ... & ... \\ \Phi_{M1} & \Phi_{M2} & ... & \Phi_{MN} \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ ... \\ u_N \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ ... \\ v_M \end{pmatrix}, \tag{2}$$

$$\Phi_{ji} = \frac{1}{y_j - x_i}, \quad i = 1, ..., N, \quad j = 1, ..., M. \tag{3}$$

and $x_1, ..., x_N$ and $u_1, ..., u_N$ are uniformly distributed on [0,1], $M = N - 1$, and each y_j is located between the closest x_i's on each side, $j = 1, ..., N - 1$ using the Single Level FMM algorithm.

1. Draw a rough sketch of the SLFMM algorithm.
2. Use the routines developed in Homework 4.
3. Evaluate the truncation number, $p(K, N)$, that provides the specified accuracy as a function of the number of boxes K and of N.
4. Write a program that implements both straightforward multiplication based on Eq. (1) and SLFMM.
5. Compare the theoretical optimal number of boxes with the actual one achieved by your program.
6. Provide a dependence of the CPU time required by the SLFMM as a function of K for $N = 5000$ $(10 < K < 200)$. Determine K_{opt} experimentally and compare with the theoretical evaluations (use the actual p). Scale $K_{opt}(N)$ for computations with varying N. Plot your scaled function $K_{opt}(N)$.
7. Provide a graph of actual error (between the standard and the fast method with $K = K_{opt}(N)$) for N varying between 10^2 and $5{\times}10^3$ and the truncation number used.
8. Provide a graph that compares the CPU time required by the straightforward method and the SLFMM for N varying between 10^2 and 10^3 for straightforward and N varying between 10^2 and 10^4 for the optimized SLFMM. Compare results with theoretical complexities of the algorithms.