Linear Algebra for Computer Vision

Introduction

CMSC 828 D

Outline

- Notation and Basics
- Motivation
- Linear systems of equations
  - Gauss Elimination, LU decomposition
- Linear Spaces and Operators
  - Addition, scalar multiplication, scalar product, transformation, operator, basis
- Eigenvalues, Eigenvectors
- Solvability conditions (“alternative theorem”)
  - Adjoint, null space, orthogonality

Outline

- Euclidean space R^3
  - distance, angles, rotations
- Metric Space
  - Distance, angles, rotations
- Least Squares
- Singular Value Decomposition
- Other Matrix decompositions

Motivation

- Fundamental to representation and numerical solution of almost all problems including those in vision and computational statistics.
  - Solving equations for calibration, stereo, tracking, …
- Geometry is fundamental to vision. However one way of doing geometry is via algebra.
  - Intersections of lines, points, planes. Determining angles. Determining orthogonal projections …
- Modern computer vision is formulated in terms of "projective geometry". Most results in projective geometry are stated algebraically and require knowledge of concepts such as rank, null space, constraints

Applications

- Rectification of images
- Calibrating cameras
- Transforming color spaces
- Tracking motion of a rigid body
- Applying constraints from multiple views
- Parametrizing fundamental matrix and trifocal tensor.

Vectors

- A vector **x** of dimension *d* represents a point in a *d* dimensional space
- Examples
  - A point in 3D Euclidean space [*x,y,z*] or 2D image space [*u,v*]
  - A point in a projective space P^3 [*X,Y,Z,W*] or in projective space P^2 [*U,V,W*]
  - Point in color space [*r,g,b*] or [*y, u, v*]
  - Point in an infinite dimensional functional space on a Fourier basis
  - Vector of intrinsic parameters for a camera (focal length, skew ratio, …)
- Essentially a short-hand notation to denote a grouping of points
  - No special structure yet

Vectors and Matrices

- d dimensional column vector **x** and its transpose

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix} \quad \text{and} \quad \mathbf{x}^t = (x_1 \; x_2 \; \ldots \; x_d)$$

- $d \times n$ dimensional matrix **M** and its transpose $\mathbf{M}^t$
- Transpose indicated with a superscript t or a prime $'$

$$\mathbf{M} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & \ldots & m_{1d} \\ m_{21} & m_{22} & m_{23} & \ldots & m_{2d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & m_{n3} & \ldots & m_{nd} \end{pmatrix} \text{and}$$

$$\mathbf{M}^t = \begin{pmatrix} m_{11} & m_{21} & \ldots & m_{n1} \\ m_{12} & m_{22} & \ldots & m_{n2} \\ m_{13} & m_{23} & \ldots & m_{n3} \\ \vdots & \vdots & \ddots & \vdots \\ m_{1d} & m_{2d} & \ldots & m_{nd} \end{pmatrix}.$$

Determinant

- Determinant of a 2x2 matrix $m_{11}m_{22}$-$m_{12}m_{21}$
- For a higher dimensional matrix we have a recursive definition

$$\begin{array}{cc} & \begin{array}{ccccccc} \phantom{m_{11}} & \phantom{m_{12}} & & j & & & \phantom{m_{1d}} \end{array} \\ \begin{array}{c} \\ \\ \\ \\ i \\ \\ \\ \end{array} & \left(\begin{array}{ccccccc} m_{11} & m_{12} & \cdots & \otimes & \cdots & \cdots & m_{1d} \\ m_{21} & m_{22} & \cdots & \otimes & \cdots & \cdots & m_{2d} \\ \vdots & \vdots & \ddots & \otimes & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \otimes & \cdots & \cdots & \vdots \\ \otimes & \otimes & \otimes & \otimes & \otimes & \otimes & \otimes \\ \vdots & \vdots & \cdots & \otimes & \cdots & \ddots & \vdots \\ m_{d1} & m_{d2} & \cdots & \otimes & \cdots & \cdots & m_{dd} \end{array} \right) \end{array} = \mathbf{M}_{i|j}. \qquad (22)$$

Given the determinants $|\mathbf{M}_{x|1}|$, we can now compute the determinant of $\mathbf{M}$ the expansion by minors on the first column giving

$$|\mathbf{M}| = m_{11}|\mathbf{M}_{1|1}| - m_{21}|\mathbf{M}_{2|1}| + m_{31}|\mathbf{M}_{3|1}| - \cdots \pm m_{d1}|\mathbf{M}_{d|1}|, \qquad (23)$$

where the signs alternate. This process can be applied recursively to the successive (smaller) matrixes in Eq. 23.

Only for a 3×3 matrix, this determinant calculation can be represented by "sweeping" the matrix, i.e., taking the sum of the products of matrix terms along a diagonal, where products from upper-left to lower-right are added with a positive sign, and those from the lower-left to upper-right with a minus sign. That is,

$$\begin{aligned} |\mathbf{M}| &= \begin{vmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{vmatrix} \qquad (24) \\ &= m_{11}m_{22}m_{33} + m_{13}m_{21}m_{32} + m_{12}m_{23}m_{31} \\ &\quad - m_{13}m_{22}m_{31} - m_{11}m_{23}m_{32} - m_{12}m_{21}m_{33}. \end{aligned}$$

Determinant: Remarks

- Determinant determines “magnitude” of matrix. Matrix with determinant =0 is called singular.
- Determinant is important in theorems
- Practically the way to compute the determinant is not this way.
- Homework problem -- determine number of operations for recursive algorithm.

Matrix basics

- Square matrix: number of rows = number of columns
- Symmetric matrix $A_{ij}=A_{ji}$.
- Skew symmetric matrix $A_{ij}=-A_{ji}$.
- Identity $\mathbf{I}_{\mathrm{ij}}= \delta_{ij}$
  - Kronecker delta $\delta_{ij}=0$ if $i\neq j$ $\delta_{ij}=1$ if $i=j$
- Lower triangular Upper triangular

$$\begin{bmatrix} a & 0 & \cdots & 0 \\ c & b & \ddots & \vdots \\ \vdots & \cdots & \ddots & 0 \\ f & g & \cdots & d \end{bmatrix} \qquad \begin{bmatrix} a & c & \cdots & d \\ 0 & b & \ddots & \vdots \\ \vdots & \ddots & \ddots & e \\ 0 & \cdots & 0 & d \end{bmatrix}$$

Matrix vector product

- $m \times n$ dimensional matrix **M** multiplies by a n dimensional vector **x** to produce a m dimensional vector

$$\begin{pmatrix} m_{11} & m_{12} & \dots & m_{1d} \\ m_{21} & m_{22} & \dots & m_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \dots & m_{nd} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ y_n \end{pmatrix},$$

$$y_j = \sum_{i=1}^{d} m_{ji} x_i.$$

Linear systems of equations

- Systems of equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

- Can be written as a matrix vector product

$$\mathbf{Ax} = \mathbf{b}$$

- Can change or scale rows
- Solved via Gauss elimination
  - Reduce system to product of lower or upper triangular matrix and **x**

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

- *O(N)* operations to solve triangular system
- *O(N^3)* operations to perform Gauss elimination

LU decomposition

- Any matrix can be written as a product of a lower triangular and upper triangular matrix.
- Most used algorithm in linear algebra.

$$\mathbf{A}=\mathbf{LU}$$

- Practically implemented by reordering equations and scaling them so that loss of accuracy is minimized.

$$\mathbf{A}=\mathbf{PLU} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- Scaled LU decomposition with "partial pivoting"
- When **A** is symmetric positive definite $x^t A x > 0$ for all x "Cholesky decomposition"

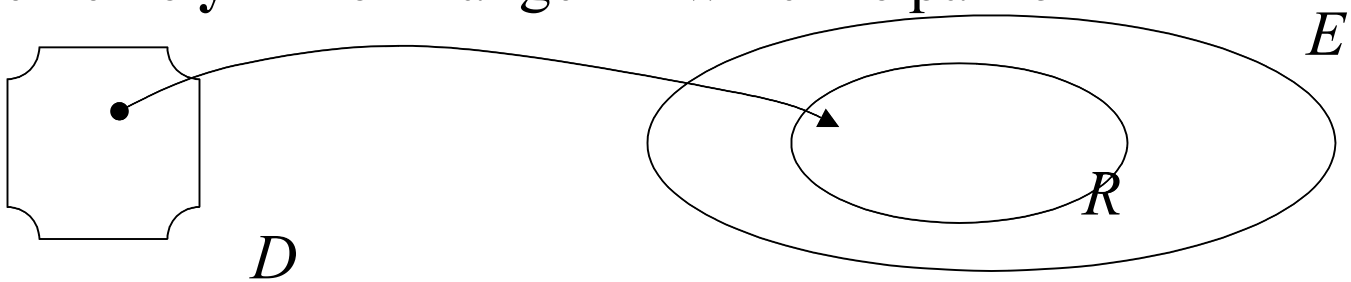
$$\mathbf{A}=\mathbf{LL}^t$$

Linear/Vector Spaces

- Previous stuff was somewhat mechanical.
- In vision we have to answer questions when
  - Models provide equations that are singular or degenerate. What can we say about the solutions? Can we restrict them?
  - Number of unknowns may be more or less than the number of observations. Can we still obtain a meaningful solution?
  - How "far" is an approximation from a solution? How do we measure this distance?
  - Matrices are operators that take one vector into another. What can we say about the properties of the operator? When is an equation involving an operator solvable?

Operators

- Function, Transformation, Operator, Mapping: synonyms
- A function takes elements **x** defined ***on*** its "Domain" *D* to elements **y** in its "Range" *R* which is part of *E*



- If for each **y** in *R* there is exactly one **x** in *D* the function is one-to-one. In this case an inverse exists whose domain is *R* and whose range is *D*
- We are interested in situations where *R* and *D* are finite-dimensional linear spaces

Vector Space

- A collection of points that obey certain rules
  - Commutative, existence of a zero element

$$u + v = v + u; \qquad u + (v + w) = (u + v) + w$$

$$\exists 0,\ u + 0 = u \ \ \forall u; \qquad u + (-u) = 0$$

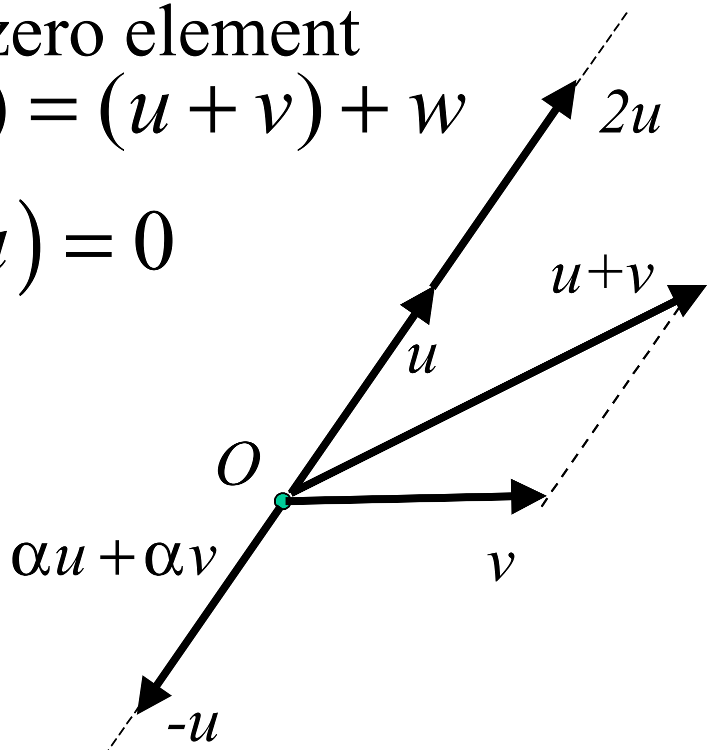
  - Scalar multiplication

$$\alpha(\beta u) = (\alpha\beta)u; \quad 1u = u$$

$$(\alpha + \beta)u = \alpha u + \beta u; \ \alpha(u + v) = \alpha u + \alpha v$$

- Let $\mathbf{u_1}, \ldots, \mathbf{u_k}$ be a set of vectors:
  Linear combination

$$\alpha_1 \mathbf{u}_1 + \cdots + \alpha_k \mathbf{u}_k$$



Manifold spanned by u

Dependence and dimensionality

- A set of vectors is dependent if for some scalars $\alpha_1, \ldots, \alpha_k$ not all zero we can write
$$\alpha_1 \mathbf{u}_1 + \cdots + \alpha_k \mathbf{u}_k = 0$$
- Otherwise the vectors are independent.
- If the zero vector is part of a set of vectors that set is dependent. If a set of vectors is dependent so is any larger set which contains it.
- A linear space is *n* dimensional if it possesses a set of n independent vectors but every *n+1* dimensional set is dependent.
- A set of vectors $\mathbf{b_1}, \ldots, \mathbf{b_k}$ is a basis for a *k* dimensional space *X* if each vector in *X* can be expressed in one and only one way as a linear combination of $\mathbf{b_1}, \ldots, \mathbf{b_k}$
- One example of a basis are the vectors (1,0,…,0), (0,1,…,0), …, (0,0, …, 1)

Distances/Metrics and Norms

- We would like to measure distances and directions in the vector space the same way that we do it in Euclidean 3D
- Distance function $d(\boldsymbol{u},\boldsymbol{v})$ makes a vector space a metric space if it satisfies
  - $d(\boldsymbol{u},\boldsymbol{v})>0$ for $\boldsymbol{u},\boldsymbol{v}$ different
  - $d(\boldsymbol{u},\boldsymbol{u})=0$, $d(\boldsymbol{u},\boldsymbol{v})=d(\boldsymbol{v},\boldsymbol{u})$
  - $d(\boldsymbol{u},\boldsymbol{w})\leq d(\boldsymbol{u},\boldsymbol{v})+d(\boldsymbol{v},\boldsymbol{w})$ *(triangle inequality)*
- Norm ("length").
  - $\|\mathbf{u}\| > 0$ for $\mathbf{u}$ not 0, $\|\mathbf{0}\|=0$
  - $\|\alpha\,\mathbf{u}\|=|\alpha|\,\|\mathbf{u}\|$, $\|\boldsymbol{u}+\boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$
- Normed linear space is a metric space with the metric defined by $d(\mathbf{u},\mathbf{v})=\|\mathbf{u}-\mathbf{v}\|$ and $\|\mathbf{u}\|=d(\mathbf{u},0)$

Dot Product

- Dot product of two vectors with same dimension

  $$\langle \mathbf{x},\mathbf{y} \rangle = \mathbf{x}^t\mathbf{y} = \sum_{i=1}^{d} x_i y_i = \mathbf{y}^t\mathbf{x}.$$
- Dot product space behaves like Euclidean R^3
- Dot product defines a norm and a metric.
- Parallelogram law

  $$||\mathbf{u}+\mathbf{v}||^2+||\mathbf{u}-\mathbf{v}||^2 = 2||\mathbf{u}||^2 + 2\,||\mathbf{v}||^2$$
- Orthogonal vectors $\langle \mathbf{u},\mathbf{v} \rangle=0$
- Angle between vectors

  $$\cos\theta=\langle \mathbf{x},\mathbf{y} \rangle/||x||\;||y||$$
- Orthonormal basis -- elements have norm 1 and are perpendicular to each other
- Other distances and products can also define a space:
  – Mahalnobis distance

Matrices as operators

- Matrix is an operator that takes a vector to another vector.
  - Square matrix takes it to a vector in the space of the same dimension.
- Dot product provides a tool to examine matrix properties
  - Adjoint matrix $< \mathbf{Au},\mathbf{v}> = <\mathbf{u},\mathbf{A}^*\mathbf{v}>$
  - Square Matrix fully defined as result of its operation on members of a basis.

$$A_{\mathrm{ij}} = < \mathbf{Ab}_{\mathrm{j}},\mathbf{b}_{\mathrm{i}}>$$

Eigenvalues and Eigenvectors

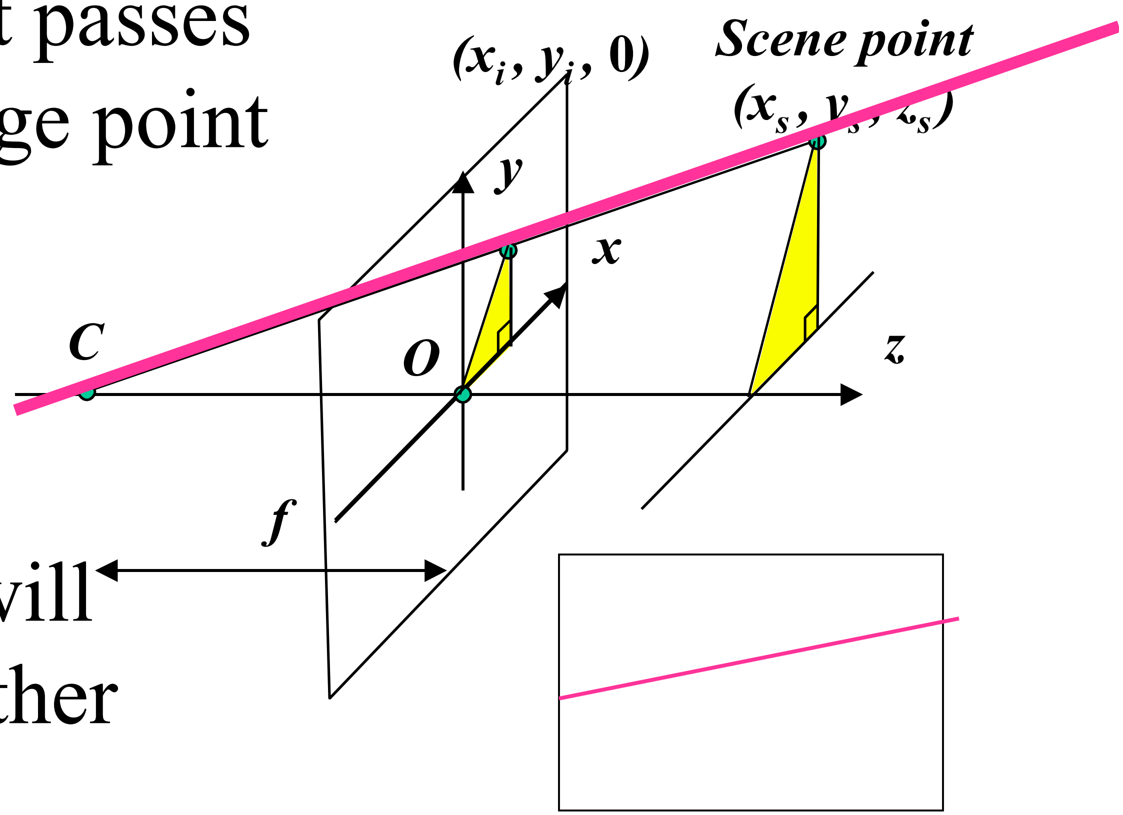
- Square matrix possesses its own natural basis.
- Eigen relation

$$\mathbf{Au}=\lambda\mathbf{u}$$

- Matrix **A** acts on vector **u** and produces a scaled version of the vector.
- Eigen is a German word meaning "proper" or "specific"
- **u** is the eigenvector while λ is the eigenvalue.
  - If **u** is an eigenvector so is $\alpha\mathbf{u}$
  - If $||\mathbf{u}||=1$ then we call it a normal eigenvector
  - λ is like a measure of the "strength" of **A** in the direction of **u**
- Set of all eigenvalues and eigenvectors of **A** is called the "spectrum of A"

Motivation: Stereo

- Point (x,y) on the image plane lies on a line in the world that passes through the image point and center of projection



•Image of this line in the world will form a line in another camera

Epipolar Constraint

- Point in one image lies on the “epipolar line” in the other image
- Algebraic statement of geometry
  - Equation of line in the other image is $\mathbf{Fm}$
  - Condition that the point $\mathbf{m}'$ lies on this line is

$$\mathbf{m}' \cdot \mathbf{Fm} = 0$$

- $\mathbf{F}$ is the “fundamental matrix”
- Estimating the fundamental matrix is an important problem in vision

$\sum m_i m_j F_{ij} = 0$

Eight point algorithm: Determining the Fundamental matrix

- Given a set of matching points in the images, Determine **F**

$$\begin{bmatrix} u' & v' & 1 \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = 0$$

$$u'\left(f_{11}u + f_{12}v + f_{13}\right) + v'\left(f_{21}u + f_{22}v + f_{23}\right) + \left(f_{31} + f_{32} + 1\right) = 0$$

Determining F

- Write expression as an equation in the unknown elements.
- If we have eight points we can solve for elements of **F,** (e.g. via LU)

$$\begin{bmatrix} u'u & u'v & u' & v'u & v'v & v' & 1 & 1 \end{bmatrix} \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \end{bmatrix} = \begin{bmatrix} -1 \end{bmatrix}$$

- **If we have more than eight points we can use a least squares formulation**