# Epipolar Geometry
# and the Fundamental Matrix

# Review about Camera Matrix **P** (from Lecture on Calibration)

- Between the world coordinates **X**=($X_s$, $X_s$, $X_s$, 1) of a scene point and the coordinates **x**=($u'$,$v'$,$w'$) of its projection, we have the following linear transformation:

with $x_{pix} = u'/w'$

$y_{pix} = v'/w'$

$$\mathbf{x} = \mathbf{P}\,\mathbf{X}$$

- P is a 3x4 matrix that completely represents the mapping from the scene to the image and is therefore called a "*camera*".

# Central Projection

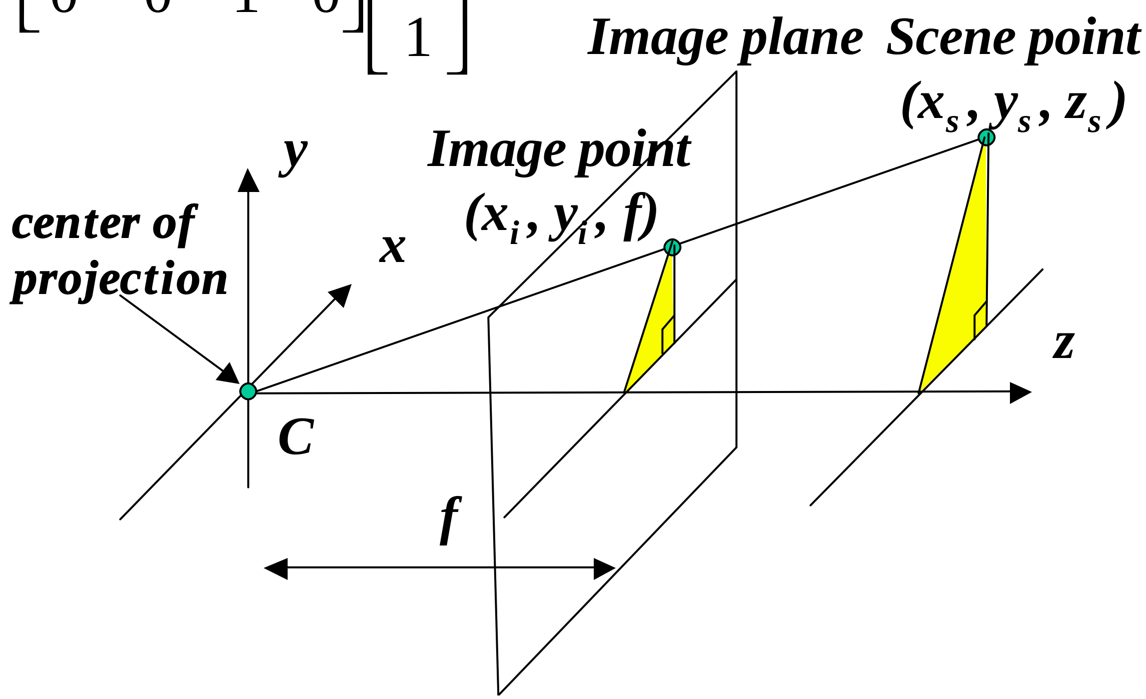
If world and image points are represented by homogeneous vectors, central projection is a linear transformation:

$$x_i = f\,\frac{x_s}{z_s}$$

$$y_i = f\,\frac{y_s}{z_s}$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_s \\ y_s \\ z_s \\ 1 \end{bmatrix}$$

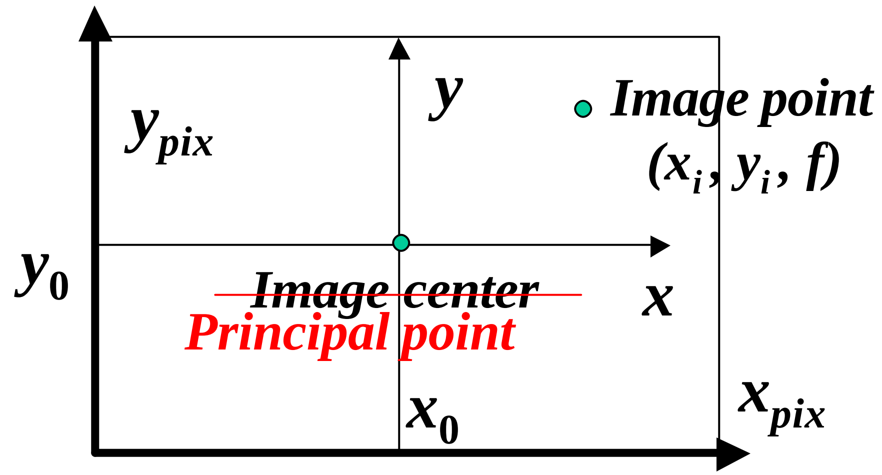
$$x_i = u / w, \quad y_i = v / w$$



# Pixel Components

Transformation uses:

- principal point $(x_0, y_0)$
- scaling factors $k_x$ and $k_y$




$$x_i = f\frac{x_s}{z_s} \qquad x_{pix} = k_x x_i + x_0 = f\,k_x\frac{x_s + z_s x_0}{z_s}$$

$$y_i = f\frac{y_s}{z_s} \qquad y_{pix} = k_y y_i + y_0 = f\,k_y\frac{y_s + z_s y_0}{z_s}$$

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} a_x & 0 & x_0 & 0 \\ 0 & a_y & y_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_s \\ y_s \\ z_s \\ 1 \end{bmatrix}$$

with $a_x = f\,k_x$, $a_y = f\,k_y$ then $x_{pix} = u'/w'$, $y_{pix} = v'/w'$

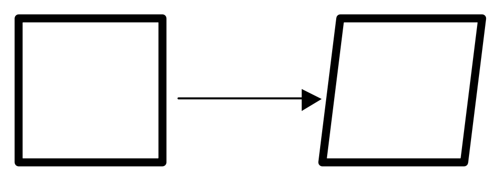
# Internal Camera Parameters

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} a_x & s & x_0 & 0 \\ 0 & a_y & y_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_s \\ y_s \\ z_s \\ 1 \end{bmatrix} \quad \text{with} \quad \begin{vmatrix} a_x = f\,k_x \\ a_y = -f\,k_y \end{vmatrix} \qquad \begin{matrix} x_{pix} = u'/w' \\ y_{pix} = v'/w' \end{matrix}$$

$$\begin{bmatrix} a_x & s & x_0 & 0 \\ 0 & a_y & y_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} a_x & s & x_0 \\ 0 & a_y & y_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \mathbf{K}\left[\mathbf{I}_3 \quad | \quad \mathbf{0}_3\right]$$

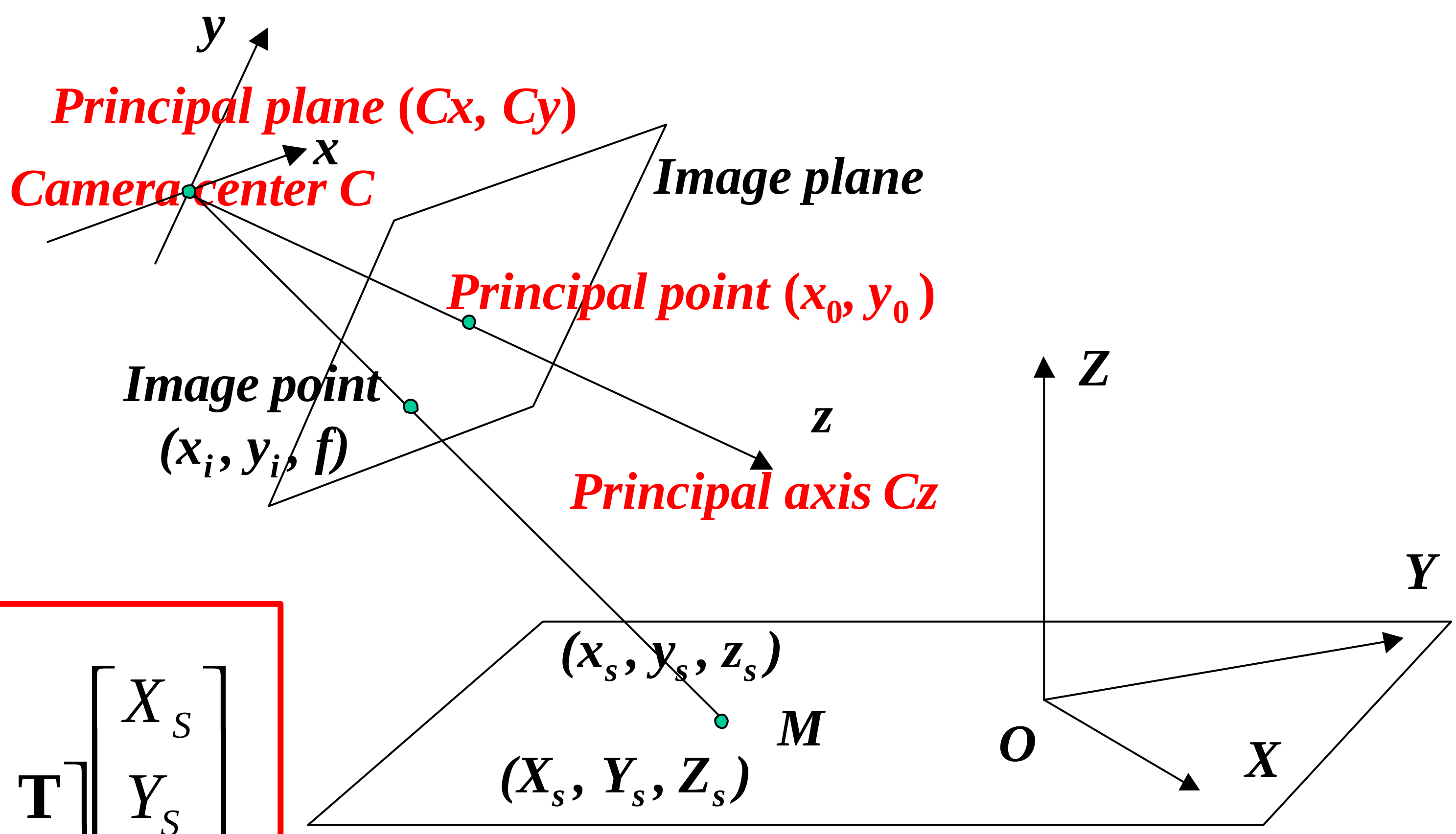
- $a_x$ and $a_y$ “focal lengths” in pixels
- $x_0$ and $y_0$ coordinates of image center in pixels

•Added parameter ***S*** is skew parameter



- **K** is called *calibration matrix*. **Five degrees of freedom.**
    - **K** is a 3x3 upper triangular matrix

# From Camera Coordinates to World Coordinates




$$\begin{bmatrix} x_S \\ y_S \\ z_S \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{T} \\ \mathbf{0}_3^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} X_S \\ Y_S \\ Z_S \\ 1 \end{bmatrix}$$

# Using Camera Center Position in World Coordinates

- We can use $-\mathbf{R}\tilde{\mathbf{C}}$ instead of $\mathbf{T}$

$$\begin{bmatrix} x_S \\ y_S \\ z_S \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & -\mathbf{R}\tilde{\mathbf{C}} \\ \mathbf{0}_3^{\mathbf{T}} & 1 \end{bmatrix} \begin{bmatrix} X_S \\ Y_S \\ Z_S \\ 1 \end{bmatrix}$$

# Linear Transformation from World Coordinates to Pixels

- Combine camera projection and coordinate transformation matrices into a single matrix **P**

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \mathbf{K}\left[\mathbf{I}_3 \quad | \quad \mathbf{0}_3\right]\begin{bmatrix} x_s \\ y_s \\ z_s \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_S \\ y_S \\ z_S \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & -\mathbf{R}\tilde{\mathbf{C}} \\ \mathbf{0}_3^{\mathbf{T}} & 1 \end{bmatrix}\begin{bmatrix} X_S \\ Y_S \\ Z_S \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \mathbf{K}\left[\mathbf{I}_3 \quad | \quad \mathbf{0}_3\right]\begin{bmatrix} \mathbf{R} & -\mathbf{R}\tilde{\mathbf{C}} \\ \mathbf{0}_3^{\mathbf{T}} & 1 \end{bmatrix}\begin{bmatrix} X_S \\ Y_S \\ Z_S \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \mathbf{P}\begin{bmatrix} X_S \\ Y_S \\ Z_S \\ 1 \end{bmatrix}$$

$$\mathbf{x} = \mathbf{P}\,\mathbf{X}$$

# Properties of Matrix **P**

- Further simplification of **P**:

$$\mathbf{x} = \mathbf{K}[\mathbf{I}_3 \quad | \quad \mathbf{0}_3]\begin{bmatrix} \mathbf{R} & -\mathbf{R}\tilde{\mathbf{C}} \\ \mathbf{0}_3^T & 1 \end{bmatrix}\mathbf{X}$$

$$[\mathbf{I}_3 \quad | \quad \mathbf{0}_3]\begin{bmatrix} \mathbf{R} & -\mathbf{R}\tilde{\mathbf{C}} \\ \mathbf{0}_3^T & 1 \end{bmatrix} = [\mathbf{R} \quad -\mathbf{R}\tilde{\mathbf{C}}] = \mathbf{R}[\mathbf{I}_3 \quad | \quad -\tilde{\mathbf{C}}]$$

$$\mathbf{x} = \mathbf{K}\mathbf{R}[\mathbf{I}_3 \quad | \quad -\tilde{\mathbf{C}}]\mathbf{X}$$

$$\boxed{\mathbf{P} = \mathbf{K}\mathbf{R}[\mathbf{I}_3 \quad | \quad -\tilde{\mathbf{C}}]}$$

- **P** has 11 degrees of freedom:
    - 5 from triangular calibration matrix **K**, 3 from **R** and 3 from $\tilde{\mathbf{C}}$
- **P** is a fairly general 3 x 4 matrix
    - left 3x3 submatrix **KR** is non-singular

# Cross-Product in Matrix Form

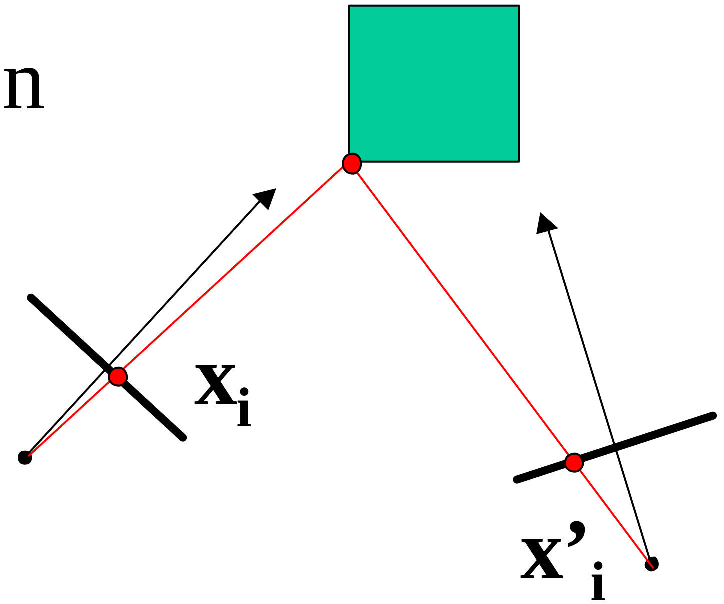
- If $\mathbf{a} = (a_1, a_2, a_3)^T$ is a 3-vector, then one can define a corresponding skew-symmetric matrix

$$[\mathbf{a}]_\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

- The cross-product of 2 vectors **a** and **b** can be written $\mathbf{a} \times \mathbf{b} = [\mathbf{a}]_\times \mathbf{b}$
- Matrix $[\mathbf{a}]_\mathbf{x}$ is singular. Its null vector (right or left) is **a**
- $\mathbf{b} \,.\, (\mathbf{a} \times \mathbf{b}) = 0 \Leftrightarrow \mathbf{b}^T [\mathbf{a}]_\times \mathbf{b} = \mathbf{0} \;\; \forall \mathbf{a}, \mathbf{b}$
- $\mathbf{c}^T [\mathbf{a}]_\times \mathbf{b} = -\; \mathbf{b}^T [\mathbf{a}]_\times \mathbf{c} \Rightarrow \mathbf{F}^T [\mathbf{a}]_\times \mathbf{F} = [\mathbf{s}]_\times \forall \mathbf{F}, \mathbf{a}$

# Definition of Epipolar Geometry

- Projective geometry between two views
- Independent of scene structure
- Depends only on the cameras' internal parameters and relative pose of cameras
- Fundamental matrix **F** encapsulates this geometry

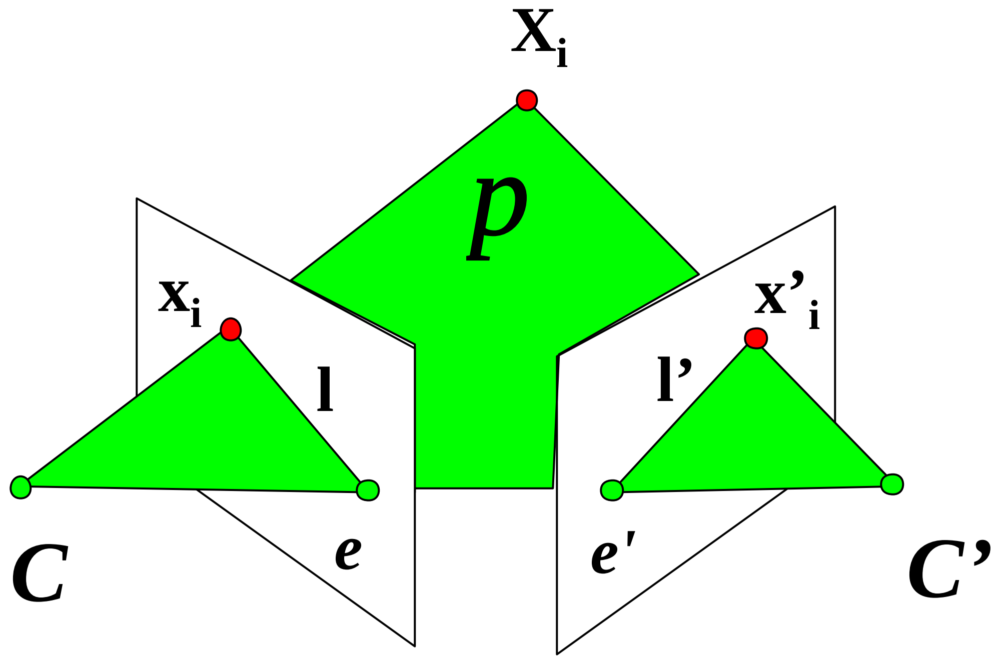



$$\mathbf{x'}_i^T \mathbf{F} \mathbf{x}_i = 0$$

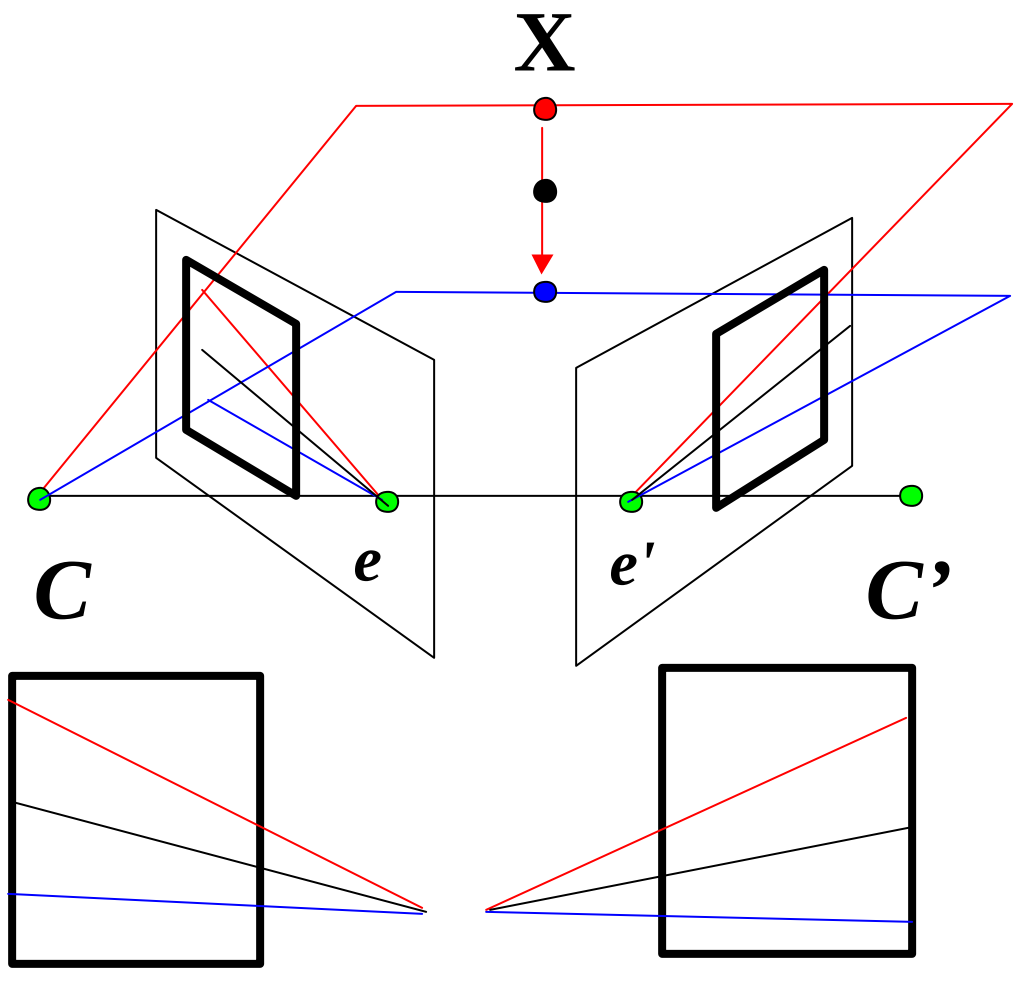
for any pair of corresponding points $\mathbf{x}_i$ and $\mathbf{x'}_i$ in the 2 images

# Relation between Image Points $x_i$ and $x_i$'

- Camera centers *C* and *C'*, scene point $\mathbf{X_i}$, image points $\mathbf{x_i}$ and $\mathbf{x'_i}$ belong to a common *epipolar plane* $p$
- *Epipoles e and e'*
  - *On baseline CC'*
- *Epipolar lines* **l** *and* **l'**

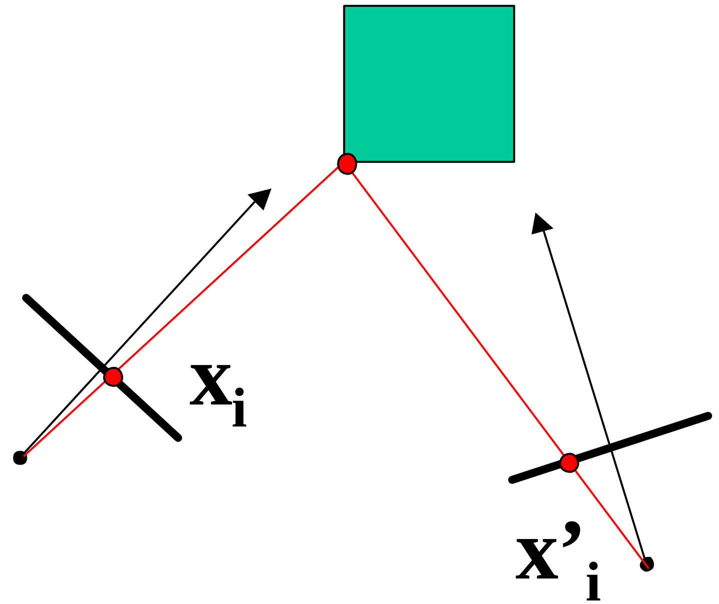


# Pencils of Epipolar Lines



# Computation of **F**

- **F** can be computed from correspondences between image points alone
- No knowledge of camera internal parameters required
- No knowledge of relative pose required

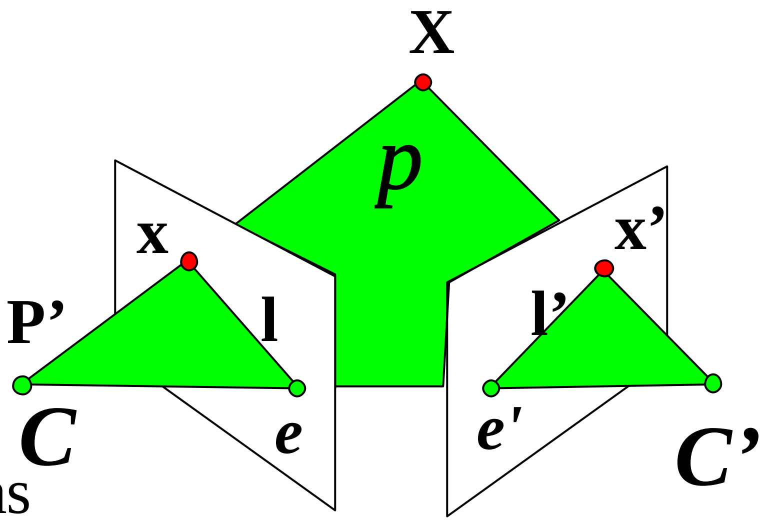



$$\mathbf{x'}_i^T \mathbf{F} \mathbf{x}_i = 0$$

for any pair of corresponding points $\mathbf{x}_i$ and $\mathbf{x'}_i$ in the 2 images

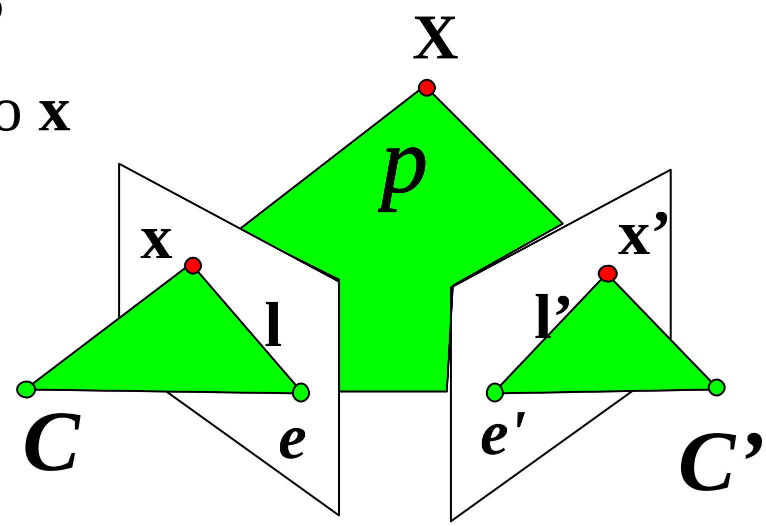
# Finding the Fundamental Matrix from Known Cameras **P** and **P'** (Outline)

- Pick up an image point **x** in camera **P**
- Find one scene point **X** on ray of **x** in camera **P**
- Find the image **x'** of **X** in camera **P'**
- Find epipole **e'** as image of **C** in camera **P'** is epipole = **P'C**
- Find epipolar line **l**' from **e' to x'** in **P'** as function of **x**
- The fundamental matrix **F** is defined by **l'**=**F x**
- **x'** belongs to **l'**, so $\mathbf{x'}^T \mathbf{l'} = 0$, so $\mathbf{x'}^T \mathbf{F} \mathbf{x} = 0$
- The fundamental matrix **F** is alternately defined by $\mathbf{x'}^T \mathbf{F} \mathbf{x} = 0$



# Finding the Fundamental Matrix from Known Cameras **P** and **P'** (Details)

- Pick up an image point **x** in camera **P**
- Find one scene point on ray from **C** to **x**
  - Point $\mathbf{X} = \mathbf{P}^{+}\mathbf{x}$ satisfies $\mathbf{x} = \mathbf{P}\mathbf{X}$
    - $\mathbf{P}^{+} = \mathbf{P}^{T}(\mathbf{P}\,\mathbf{P}^{T})^{-1}$, so $\mathbf{P}\mathbf{X} = \mathbf{P}\,\mathbf{P}^{T}(\mathbf{P}\,\mathbf{P}^{T})^{-1}\mathbf{x} = \mathbf{x}$




- Image of this point in camera **P'** is $\mathbf{x'} = \mathbf{P'}\mathbf{X} = \mathbf{P'}\,\mathbf{P}^{+}\mathbf{x}$
- Image of **C** in camera **P'** is epipole $\mathbf{e'} = \mathbf{P'}\mathbf{C}$
- Epipolar line of **x** in **P'** is

$$\mathbf{l'} = (\mathbf{e'}) \times (\mathbf{P'}\mathbf{P}^{+}\mathbf{x}) = [\mathbf{e'}]_{\times}\mathbf{P'}\mathbf{P}^{+}\mathbf{x}$$

- $\mathbf{l'} = \mathbf{F}\,\mathbf{x}$ defines **F** fundamental matrix $\Rightarrow$ $\mathbf{F} = [\mathbf{P'}\mathbf{C}]_{\times}\mathbf{P'}\mathbf{P}^{+}$
- **x'** belongs to **l'**, so $\mathbf{x'}^{T}\mathbf{l'} = 0$, so $\mathbf{x'}^{T}\mathbf{F}\,\mathbf{x} = 0$

# Properties of Fundamental Matrix F

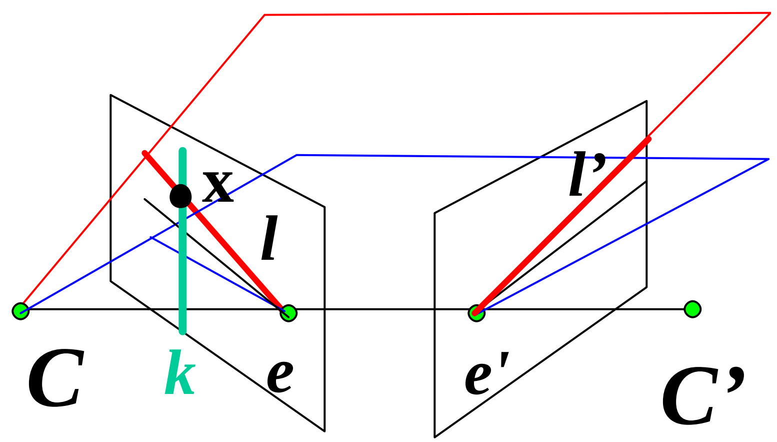
- Matrix 3X3 (since $\mathbf{x'}^T \mathbf{F} \mathbf{x} = \mathbf{0}$ )
- If **F** is fundamental matrix of camera pair (**P, P'**) then the fundamental matrix **F'** of camera pair (**P', P**) is equal to $\mathbf{F}^T$
  - $\mathbf{x}^T \mathbf{F'} \mathbf{x'} = \mathbf{0}$ implies $\mathbf{x'}^T \mathbf{F'}^T \mathbf{x} = \mathbf{0}$, so $\mathbf{F'} = \mathbf{F}^T$
- Epipolar line of **x** is $\mathbf{l'} = \mathbf{F} \mathbf{x}$
- Epipolar line of **x'** is $\mathbf{l} = \mathbf{F}^T \mathbf{x'}$

# More Properties of **F**

- Epipole **e'** is left null space of **F**, and **e** is right null space.
  - All epipolar lines **l'** contains epipole **e'**, so $\mathbf{e'}^T \mathbf{l'} = \mathbf{0}$, i.e. $\mathbf{e'}^T \mathbf{F}\,\mathbf{x} = \mathbf{0}$ for all **x.** Therefore $\mathbf{e'}^T \mathbf{F} = \mathbf{0}$
    Similarly $\mathbf{e}^T \mathbf{F}^T \mathbf{x'} = \mathbf{0}$ implies $\mathbf{e}^T \mathbf{F}^T = \mathbf{0}$, therefore $\mathbf{F}\,\mathbf{e} = \mathbf{0}$
- **F** is of rank 2 because $\mathbf{F} = [\mathbf{e'}]_{\mathbf{x}} \mathbf{P'}\mathbf{P}^{+}$ and $[\mathbf{e'}]_{\mathbf{x}}$ is of rank 2
- **F** has 7 degrees of freedom
  - There are 9 elements, but scaling is not significant
  - Det **F** = 0 removes one degree of freedom

# Mapping between Epipolar Lines (a Homography)

- Define **x** as intersection between line **l** and a line **k** (**k** does not pass through epipole **e**):




$$\mathbf{x} = \mathbf{k} \times \mathbf{l} = [\mathbf{k}]_{\mathbf{x}}\, \mathbf{l}$$

$$\mathbf{l}' = \mathbf{F}\,\mathbf{x} = \mathbf{F}[\mathbf{k}]_{\mathbf{x}}\, \mathbf{l}$$

- Line **e** does not pass through point **e**

$$\mathbf{l}' = \mathbf{F}\,\mathbf{x} = \mathbf{F}[\mathbf{e}]_{\mathbf{x}}\, \mathbf{l}$$

- Similarly

$$\mathbf{l} = \mathbf{F}^{\mathrm{T}}\,\mathbf{x}' = \mathbf{F}^{\mathrm{T}}[\mathbf{e}']_{\mathbf{x}}\, \mathbf{l}'$$

# Retrieving Camera Matrices **P** and **P'** from Fundamental Matrix **F**

- General form of **P** is $\mathbf{P} = \mathbf{K}\,\mathbf{R}\left[\mathbf{I}_3 \quad | \quad -\tilde{\mathbf{C}}\right]$
- Select world coordinates as camera coordinates of first camera, select focal length = 1, and count pixels from the principal point. Then $\mathbf{P} = [\ \mathbf{I}_3 \mid \mathbf{0}]$
- Then $\mathbf{P'} = [\mathbf{S\,F} \mid \mathbf{e'}]$ with **S** any skew-symmetric matrix is a solution. Proof:
  - $\mathbf{x'}^{T}\,\mathbf{F\,x} = \mathbf{X}^{T}\,\mathbf{P'}^{T}\,\mathbf{F\,P\,X}$
  - $\mathbf{P'}^{T}\,\mathbf{F\,P} = [\mathbf{S\,F} \mid \mathbf{e'}]^{T}\,\mathbf{F}\,[\ \mathbf{I}_3 \mid \mathbf{0}\ ]$ is skew-symmetric
  - For any skew-symmetric matrix **S'** and any **X**, $\mathbf{X}^{T}\,\mathbf{S'}\,\mathbf{X} = \mathbf{0}$
- $\mathbf{S} = [\mathbf{e'}]_{\mathbf{x}}$ is a good choice. Therefore $\mathbf{P'} = [[\mathbf{e'}]_{\mathbf{x}}\,\mathbf{F} \mid \mathbf{e'}]$

# $[\mathbf{S}\ \mathbf{F}\ |\ \mathbf{e'}]^T\ \mathbf{F}\ [\ \mathbf{I}_3\ |\ \mathbf{0}\ ]$ is skew-symmetric

$$[\mathbf{SF}\,|\,\mathbf{e'}]^T\ \mathbf{F}\ [\mathbf{I}_3\,|\,\mathbf{0}_3] = \begin{bmatrix} \mathbf{F}^T\mathbf{S}^T\mathbf{F} & \mathbf{0}_3 \\ \mathbf{e'}^T\mathbf{F} & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{F}^T\mathbf{S}^T\mathbf{F} & \mathbf{0}_3 \\ \mathbf{0}_3^{\ T} & 0 \end{bmatrix}$$

- $\mathbf{e'}^T\mathbf{F}=\mathbf{0}$ because $\mathbf{e'}$ is left null space of $\mathbf{F}$
- $\mathbf{F}^T\mathbf{S}^T\mathbf{F}$ is skew-symmetric for any $\mathbf{F}$ and any skew-symmetric $\mathbf{S}$

# Essential Matrix $\mathbf{E}$

- Specialization of fundamental matrix for calibrated cameras and normalized coordinates
  - $\mathbf{x = P\ X = K\ [R \mid T]\ X}$
  - Normalize coordinates: $\mathbf{x_0 = K^{-1}\ x = [R \mid T]\ X}$
- Consider pair of normalized cameras
  - $\mathbf{P = [I \mid 0],\ \ P' = [R \mid T]}$
- Then we compute $\mathbf{F} = [\mathbf{P'C}]_\times \mathbf{P'P}^+$

$$[\mathbf{P'C}]_\times = [[\mathbf{R} \mid \mathbf{T}][\mathbf{0\ 0\ 0\ 1}]^{\mathbf{T}}]_\times = [\mathbf{T}]_\times$$

$$\mathbf{P}^+ = \begin{bmatrix} \mathbf{I_3} \\ \mathbf{0_3^T} \end{bmatrix},\ \mathbf{P'P}^+ = \mathbf{R} \Rightarrow \boxed{\mathbf{F} = [\mathbf{T}]_\times \mathbf{R} \equiv \mathbf{E}}$$

# Essential Matrix and Fundamental Matrix

- The defining equation for essential matrix is $\mathbf{x_0'^T E x_0 = 0}$, with
  - $\mathbf{x_0 = K^{-1} x}$
  - $\mathbf{x_0' = K'^{-1} x'}$
- Therefore $\mathbf{x'^T K'^{-T} E K^{-1} x = 0}$
- Comparing with $\mathbf{x'^T F x = 0}$, we get

$$\mathbf{E = K'^T F K}$$

# Computing Fundamental Matrix from Point Correspondences

- The fundamental matrix is defined by the equation $\mathbf{x'}_{\mathbf{i}}^{\mathbf{T}} \mathbf{F} \mathbf{x}_{\mathbf{i}} = 0$ for any pair of corresponding points $\mathbf{x}_{\mathbf{i}}$ and $\mathbf{x'}_{\mathbf{i}}$ in the 2 images
- The equation for a pair of points $(x, y, 1)$ and $(x', y', 1)$ is:

$$x'x\, f_{11} + x'y\, f_{12} + x'\, f_{13} + \\ + y'x\, f_{21} + y'y\, f_{22} + y'\, f_{23} + \\ + x\, f_{31} + y\, f_{32} + f_{33} = 0$$

- For $n$ point matches:

$$\mathbf{A}\,\mathbf{f} = \begin{bmatrix} x'_1 x_1 & x'_1 y_1 & x'_1 & y'_1 x_1 & y'_1 y_1 & y'_1 & x_1 & y_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x'_n x_n & x'_n y_n & x'_n & y'_n x_n & y'_n y_n & y'_n & x_n & y_n & 1 \end{bmatrix} \mathbf{f} = 0$$

# Computing Fundamental Matrix from Point Correspondences

- We have a homogeneous set of equations
  $\mathbf{A}\,\mathbf{f} = 0$
- **f** can be determined only up to a scale, so there are 8 unknowns, and at least 8 point matchings are needed
  - hence the name “8 point algorithm”
- The least square solution is the singular vector corresponding the smallest singular value of **A**, i.e. the last column of **V** in the SVD $\mathbf{A} = \mathbf{U}\,\mathbf{D}\,\mathbf{V}^T$

# Next Class

- 3D Reconstruction from Multiple Views

# References


- Multiple View Geometry in Computer Vision, R. Hartley and A. Zisserman, Cambridge University Press, 2000.