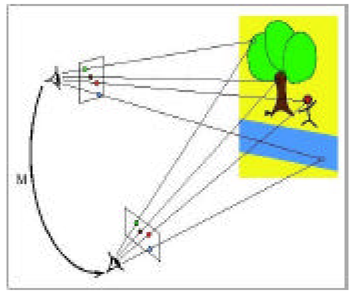
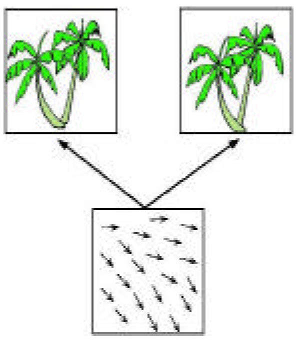
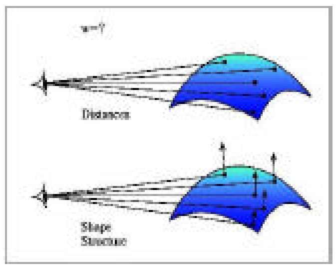
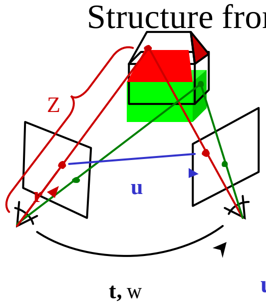
Motion and Flow II







Structure from Motion

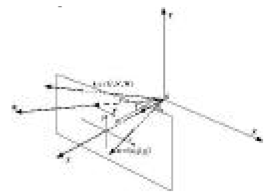


$$\mathbf{u} = \mathbf{u}_{tr} + \mathbf{u}_{rot}$$

$$\mathbf{u}_{tr} = \frac{1}{Z}\left(\hat{\mathbf{z}} \times \left(\mathbf{t} \times \mathbf{r}\right)\right)$$

$$\mathbf{u}_{rot} = \frac{1}{F}\left(\hat{\mathbf{z}} \times \left(\mathbf{r} \times \left(\left[? \right]_{\times} \mathbf{r}\right)\right)\right)$$

Passive Navigation and Structure



Thesystemmoveswith a rigid motion with translaitonal velocity $\mathbf{t} = (U, V, W)^T$ and rotationalvelocityw $= (a, b, g)^T$.
Scenepoints $\mathbf{R} = (X, Y, Z)^T$ projectonto imagepoints $\mathbf{r} = (x, y, f)$
and the 3D velocity $\dot{\mathbf{R}} = (U, V, W)$ of a scenepoint is observedin the image as velocity $\dot{\mathbf{r}} = (u, v.0)$.

Image Flow due to Rigid Motion

The velocity of a point with respect to the *XYZ* coordinate system is

$$\dot{\mathbf{R}} = -\mathbf{t} - \mathrm{w} \times \mathbf{R}$$

$$\dot{X} = -U - BZ + gY$$

$$\dot{Y} = -V - gX + aZ$$

$$\dot{Z} = -W - aX + bX$$

Let $f = 1$, then $x = \frac{X}{Z} \quad y = \frac{Y}{Z}; \quad u = \dot{x} \quad v = \dot{y}$

$$u = \left(\frac{\dot{X}}{Z}\right) = \frac{\dot{X}}{Z} - \frac{X\dot{Z}}{Z^2} = \left(-\frac{U}{Z} - b + gy\right) - x\left(-\frac{W}{Z} - ay + bx\right)$$

$$v = \left(\frac{\dot{Y}}{Z}\right) = \frac{\dot{Y}}{Z} - \frac{Y\dot{Z}}{Z^2} = \left(-\frac{V}{Z} - gx + a\right) - y\left(-\frac{W}{Z} - ay + bx\right)$$

$$u = \frac{-U + xW}{Z} + axy - b\left(x^2 + 1\right) + gy = \frac{u_{tr}}{Z} + u_{rot}$$

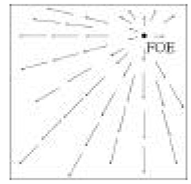
$$v = \frac{-V + yW}{Z} + a\left(y^2 + 1\right) - bxy - gx = \frac{v_{tr}}{Z} + v_{rot}$$

in vector notation: $\mathbf{r} = \frac{1}{\mathbf{R} \cdot \mathbf{z}_0}\mathbf{R}$, where $Z = \mathbf{R} \cdot \mathbf{z}_0$

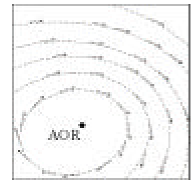
$$\dot{\mathbf{r}} = \frac{1}{\mathbf{R} \cdot \mathbf{z}_0}\left(\mathbf{z}_0 \times (\mathbf{t} \times \mathbf{r})\right) + \mathbf{z}_0 \times \left(\mathbf{r} \times (\mathrm{w} \times \mathbf{r})\right)$$

Scaling ambiguity: We can compute the translation only up to a scale factor ($K\mathbf{t}$, KZ) give the same flow as ($\mathbf{t}$, Z).

Translational flow field



Rotational flow field



$$\frac{\mathbf{u}_{tr}}{Z} = \left((x - x_0)\frac{W}{Z}, (y - y_0)\frac{W}{Z}\right)$$

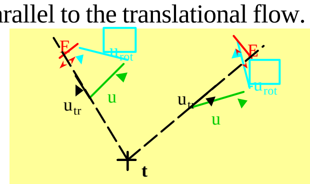
where $(x_0, y_0) = \left(\frac{U}{W} \cdot f, \frac{V}{W} \cdot f\right)$ is the focus of expansion (FOE)

or focus of contraction (FOC).

$\left(\frac{a}{g} f, \frac{b}{g} f\right)$ is the point where the rotation axis pierces the image plane (AOR).

Classical Structure from Motion

- Established approach is the epipolar minimization: The "derotated flow" should be parallel to the translational flow.



Uniqueness

Let there be two translations $\mathbf{t}_1, \mathbf{t}_2$ and two surfaces Z_1, Z_2

$$\mathbf{t}_1 = (U_1, V_1, W_1) \qquad \mathbf{t}_2 = (U_2, V_2, W_2)$$

$$u = \frac{-U_1 + xW_1}{Z_1} \qquad v = \frac{-V_1 + yW_1}{Z_1}$$

$$u = \frac{-U_2 + xW_2}{Z} \qquad v = \frac{-V_2 + yW_2}{Z}$$

$$\frac{-U_1 + xW_1}{-V_1 + yW_1} = \frac{-U_2 + xW_2}{-V_2 + yW_2}$$

$$(-U_1 + xW_1)(-V_2 + yW_2) = (-U_2 + xW_2)(-V_1 + yW_1)$$

$$U_1V_2 - xV_2W_1 - yU_1W_2 + xyW_1W_2 = U_2V_1 - xV_1W_2 - yU_2W_1 + xyW_2W_1$$

must hold for all x and y

$$\begin{array}{lcl} U_1V_2 & = & U_2V_1 \\ V_2W_1 & = & V_1W_2 \\ U_1W_2 & = & U_2W_1 \end{array} \rightarrow \begin{array}{c} U_1 : V_1 : W_1 = U_2 : V_2 : W_2 \\ Z_2 = KZ_1 \end{array} \rightarrow \mathbf{t}_2 = k\mathbf{t}_1 \text{ and } Z_2 = kZ_1$$

A translational flow field determines the motions of the camera uniquely up to a scaling factor.

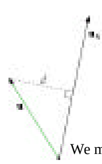
The Translational Case

A least squares formulation

$$\iint \left(u - \frac{xW - U}{Z}\right)^2 + \left(v - \frac{yW - V}{Z}\right)^2 dx\, dy \rightarrow \min$$

Substitute $a = -U + xW \quad b = -V + yW$

$$\iint \left(u - \frac{a}{Z}\right)^2 + \left(v - \frac{b}{Z}\right)^2 \rightarrow \min$$



We minimize d^2

Step 1: Minimize with respect to Z. (Find the length of $\mathbf{u}_{tr}$ for which d would be minimized)

$$\left(u - \frac{a}{Z}\right)\frac{a}{Z^2} + \left(v - \frac{b}{Z}\right)\frac{b}{Z^2} = 0 \qquad Z = \frac{a^2 + b^2}{ua + vb}$$

Substitute back

$$\iint \frac{(ub - va)^2}{a^2 + b^2} dx\, dy \rightarrow \min$$

Step 2: Differentiate with respect to U, V, W, set expression to zero.

Let $K = \frac{(ub - va)(ua + vb)}{\left(a^2 + b^2\right)^2}$

$$\text{I} \iint (-V + yW) K \, dx\, dy = 0$$

$$\text{II} \iint (-U + xW) K \, dx\, dy = 0$$

$$\text{III} \iint (-yU + xV) K \, dx\, dy = 0$$

3 linearly dependent equations $(U \cdot \text{I} + V \cdot \text{II} + W \cdot \text{III} = 0)$

The Rotational Case

$$\iint (u-u_{rot})^2 + (v-v_{rot})^2 \rightarrow \min$$

$$u - axy + b(x^2+1) - gy = 0$$

$$v - a(y^2+1) + bxy + gx = 0$$

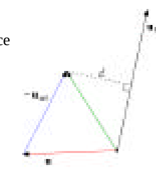
$$\begin{pmatrix} xy & -(x^2+1) & y \\ (y^2+1) & -xy & -x \end{pmatrix} \begin{pmatrix} a \\ b \\ g \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

In matrix form $A \cdot \mathrm{w} = \mathbf{u}$

$$\mathrm{w} = (A^T A)^{-1} A^T \mathbf{u}$$

The General Case

Minimization of epipolar distance



$$\iint \left(\begin{pmatrix} u-u_{rot} \\ v-v_{rot} \end{pmatrix} \cdot \begin{pmatrix} -v_{tr} \\ u_{tr} \end{pmatrix} \right)^2 dxdy \rightarrow \min$$

or, in vector notation

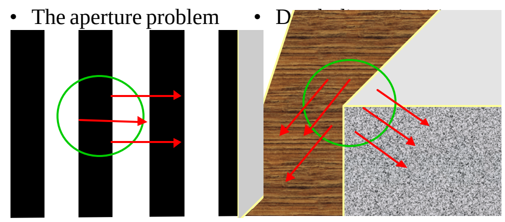
$$\int ((\mathbf{t}\times\mathbf{r})(\dot{\mathbf{r}} - \mathrm{w}\times\mathbf{r}))^2 d\mathbf{r} \rightarrow \min$$

Motion Estimation Techniques

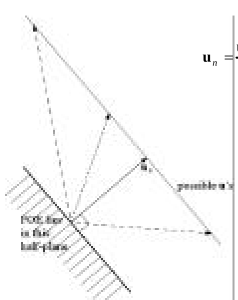
- Linearization (Tsai Huang 1984, Longuet-Higgins 1981) in the discrete case
  $\mathbf{x}'E\mathbf{x} = 0$, where $E = T_x\mathbf{R}$
  $\mathbf{x}'E\mathbf{x} = \mathbf{a}^T e$
  with $a = (x'x, x'y, x', y'x, y'y, y', x, y, 1)$
  1. LS minimization $\sum (a^T e)^2 \rightarrow \min$ solve for E.
  2. Obtain from E translation and rotation using SVD.
- Prazdny (1981), Burger Bhanu (1990), Nelson Aloimonos (1988), Heeger Jepson (1992): Decomposition of flow field into translational and rotational components. Translational flow field has a certain structure: All vectors are emanating from a point. Either search in the space of rotations or the space of translational directions.
- Longuet-Higgins Prazdny (1980), Waxman (1987): Parametric model for local surface patches ⌫ solve locally for motion parameters and structure

Optical flow difficulties

- The aperture problem
- D



Translational Normal Flow



$$\mathbf{u}_n = \frac{\mathbf{u}_{tr}}{Z} \cdot \mathbf{n}$$

- In the case of translation each normal flow vector constrains the location of the FOE to a half-plane.
- Intersection of half-planes provides FOE.

Egoestimation from normal flow

- patterns defined on the sign of normal flow along particular orientation fields
- positive depth constraint
- 2 classes of orientation fields: copoint vectors and coaxis vectors

Copoint vector fields

Copoint vectors: $\mathbf{v}_{cp}(\mathbf{t}_i)$ perpendicular to translational flow field defined by $\mathbf{t}_i$

$\mathbf{v}_{cp}(\mathbf{t}_i) = \hat{\mathbf{z}} \times \mathbf{u}_{tr}(\mathbf{t}_i) = \hat{\mathbf{z}} \times (\hat{\mathbf{z}} \times (\mathbf{t}_i \times \mathbf{r}))$

The components of flow along $\mathbf{v}_{cp}(\mathbf{t}_i)$ amount to

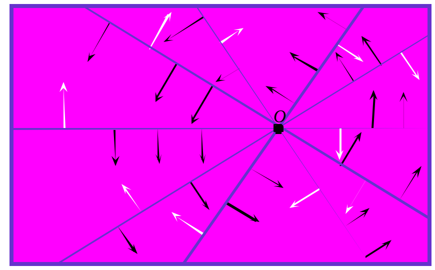
$$\dot{\mathbf{r}} \cdot \frac{\mathbf{v}_{cp}}{\|\mathbf{v}_{cp}\|} = \frac{1}{\|\mathbf{v}_{cp}\|}\left(\frac{1}{Z}(\mathbf{t} \times \mathbf{t}_i) \cdot \mathbf{r} + (w \times \mathbf{r}) \cdot (\mathbf{t}_i \times \mathbf{r})\right)$$

Thus the translational component is separated by a line $(\mathbf{t} \times \mathbf{t}_i) \cdot \mathbf{r} = 0$ into positive and negative values

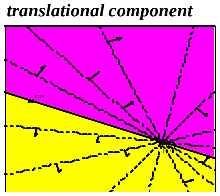
The rotational component is separated by a second-order curve $(w \times \mathbf{r}) \cdot (\mathbf{t}_i \times \mathbf{r}) = 0$ into positive and negative values

Pattern with positive areas, negative areas, and some undefined areas

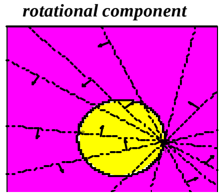
copoint vectors

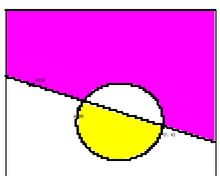


translational component



rotational component





Coaxis vector fields

Coaxis vectors: $\mathbf{v}_{ca}(w_?)$ perpendicular to rotation

$$\mathbf{v}_{ca}(w_?) = \hat{\mathbf{z}} \times \mathbf{u}_{rot}(w_?) = \hat{\mathbf{z}} \times (\times \mathbf{r} \times (w_? \times \mathbf{r}))$$

The components of flow along $\mathbf{v}_{ca}(w_i)$ amount to

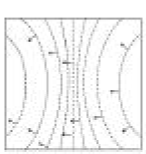
$$\dot{\mathbf{r}} \cdot \frac{\mathbf{v}_{ca}}{\|\mathbf{v}_{ca}\|} = \frac{1}{\|\mathbf{v}_{ca}\|}\left((w \times w_i) \cdot \mathbf{r} + \frac{1}{Z}(\mathbf{t} \times \mathbf{r}) \cdot (w_i \times \mathbf{r})\right)$$

Thus the translational component is separated by a second-order curve $(\mathbf{t} \times \mathbf{r}) \cdot (w_? \times \mathbf{r}) = 0$

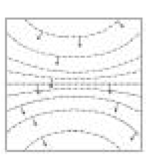
and the rotational component is separated by a line $(w \times w_?) \cdot \mathbf{r} = 0$

Intersection of patterns provides the FOE.

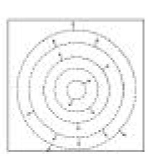
Three coaxis vector fields



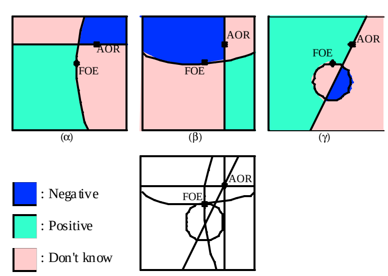
(a)

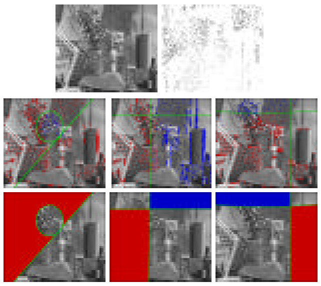


(b)



(c)





Depth variability constraint

- Errors in motion estimates lead to distortion of the scene estimates.
- The distortion is such that the correct motion gives the "smoothest" (least varying) scene structure.

Depth estimation

- Scene depth can be estimated from normal flow measurements:

$$u_n = \mathbf{u}\cdot\mathbf{n} = \frac{1}{Z}\mathbf{u}_{tr}\cdot\mathbf{n} + \mathbf{u}_{rot}\cdot\mathbf{n}$$

$$\frac{1}{\hat{Z}} = \frac{u_n - \mathbf{u}_{rot}(\hat{?})\cdot\mathbf{n}}{\mathbf{u}_{tr}(\hat{\mathbf{t}})\cdot\mathbf{n}}$$

Visual Space Distortion

$$\hat{Z} = Z \cdot D, \quad D = \frac{\mathbf{u}_{tr}(\hat{\mathbf{t}})\cdot\mathbf{n}}{[\mathbf{u}_{tr}(\mathbf{t}) - \mathbf{u}_{rot}(d?)]\cdot\mathbf{n}}$$

- Wrong 3D motion gives rise to a rugged (unsmooth) depth function (surface).
- The correct 3D motion leads to the "smoothest" estimated depth.

The error function

- A normal flow measurement:

$$u_n = \frac{1}{Z}\mathbf{u}_{tr}\cdot\mathbf{n} + \mathbf{u}_{rot}\cdot\mathbf{n}$$

- The error function to be minimized:

$$\Theta = \sum_R \sum_i W_i (\hat{u}_n - u_n)^2$$

- Global parameters: $\hat{\mathbf{t}}, \hat{?}$
- Local parameter: $\hat{Z}$

Error function evaluation

- Given a translation candidate $\hat{\mathbf{t}}$, each local depth can be computed as a linear function of the rotation $\hat{?}$.
- We obtain a second order function of the rotation; its minimization provides both the rotation and the value of the error function.

Handling depth discontinuities

- Given a candidate motion, the scene depth can be estimated and further processed to find depth discontinuities.
- Split a region if it corresponds to two depth values separated in space.

The algorithm

- Compute spatio-temporal image derivatives and normal flow.
- Find the direction of translation that minimizes the depth-variability criterion.
  - Hierarchical search of the 2D space.
  - Iterative minimization.
  - Utilize continuity of the solution in time; usually the motion changes slowly over time.

Sources:

- Horn (1986)
- http://www.cfar.umd.edu/ftp/TRs/CVL-Reports-1993/TR3064-Fermuller.ps.gz
  http://www.cfar.umd.edu/ftp/Trs/CVL-Reports-1995/TR3484-Fermuller.ps.gz (patterns on normal flow)
- http://www.cfar.umd.edu/ftp/TRs/CVL-Reports-1999/TR4000-brodsky.ps.gz (depth variability constraint)