Solutions to Homework 11

1. (a) If we have a motion $\{\mathbf{t}_1, \omega_1\}$ and a surface $Z_1(x, y)$ that yields the same motion field as a motion $\{\mathbf{t}_2, \omega_2\}$ and a surface $Z_2(x, y)$, then

$$\frac{1}{Z_1}(\hat{\mathbf{z}}\times(\mathbf{t}_1\times\mathbf{r})) + \hat{\mathbf{z}}\times(\mathbf{r}\times(\omega_1\times\mathbf{r})) = \frac{1}{Z_2}(\hat{\mathbf{z}}\times(\mathbf{t}_2\times\mathbf{r})) + \hat{\mathbf{z}}\times(\mathbf{r}\times(\omega_2\times\mathbf{r}))$$

or
$$\frac{1}{Z_1}(\hat{\mathbf{z}}\times(\mathbf{t}_1\times\mathbf{r})) - \frac{1}{Z_2}(\hat{\mathbf{z}}\times(\mathbf{t}_2\times\mathbf{r})) = \hat{\mathbf{z}}\times(\mathbf{r}\times(\delta\omega\times\mathbf{r})) \quad (1)$$

with $\delta\omega = \omega_2 - \omega_1$.

We only consider the general case. Thus let

$$\mathbf{t}_1\cdot\mathbf{t}_2 \neq 0, \quad \delta\omega\cdot\mathbf{t}_1 \neq 0, \quad \delta\omega\cdot\mathbf{t}_2 = 0$$
$$||\mathbf{t}_1\times\mathbf{t}_2|| \neq 0, \quad ||\delta\omega\times\mathbf{t}_1|| \neq 0, \quad ||\delta\omega\times\mathbf{t}_2|| \neq 0$$

To find the surface Z_1, we take the dot product of (1) with $\mathbf{t}_2\times\mathbf{r}$ to obtain

$$\frac{1}{Z_1}(\hat{\mathbf{z}}\times(\mathbf{t}_1\times\mathbf{r}))\cdot(\mathbf{t}_2\times\mathbf{r}) = (\hat{\mathbf{z}}\times(\mathbf{r}\times(\delta\omega\times\mathbf{r})))\cdot(\mathbf{t}_2\times\mathbf{r})$$

or
$$\frac{1}{Z_1}((\hat{\mathbf{z}}\cdot\mathbf{r})\,\mathbf{t}_1 - (\hat{\mathbf{z}}\cdot\mathbf{t}_1)\,\mathbf{r})\cdot(\mathbf{t}_2\times\mathbf{r}) = ((\hat{\mathbf{z}}\cdot(\delta\omega\times\mathbf{r}))\,\mathbf{r} - (\hat{\mathbf{z}}\cdot\mathbf{r})\,(\delta\omega\times\mathbf{r}))\cdot(\mathbf{t}_2\times\mathbf{r})$$

or
$$\frac{1}{Z_1}(\hat{\mathbf{z}}\cdot\mathbf{r})\,\mathbf{t}_1\cdot(\mathbf{t}_2\times\mathbf{r}) = (\mathbf{r}\times\delta\omega)\cdot(\mathbf{t}_2\times\mathbf{r})$$

or
$$\frac{1}{Z_1}(\mathbf{t}_1\times\mathbf{t}_2)\cdot\mathbf{r} = (\mathbf{r}\times\delta\omega)\cdot(\mathbf{t}_2\times\mathbf{r}) \quad (2)$$

Similarly, by taking the dot product of (1) with $(\mathbf{t}_1\times\mathbf{r})$ we obtain an equation for the surface Z_2

$$\frac{1}{Z_2}(\mathbf{t}_1\times\mathbf{t}_2)\cdot\mathbf{r} = (\mathbf{r}\times\delta\omega)\cdot(\mathbf{t}_1\times\mathbf{r}) \quad (3)$$

To express (2) and (3) in scene coordinates we substitute $Z_1\mathbf{r} = \mathbf{R}$ into (2) to obtain

$$(\mathbf{t}_1\times\mathbf{t}_2)\cdot\mathbf{R} + (\delta\omega\times\mathbf{R})\cdot(\mathbf{t}_2\times\mathbf{R}) = 0$$

Similarly, substituting $Z_2\mathbf{r} = \mathbf{R}$ into (3) we obtain

$$(\mathbf{t}_1\times\mathbf{t}_2)\cdot\mathbf{R} + (\delta\omega\times\mathbf{R})\cdot(\mathbf{t}_1\times\mathbf{R}) = 0$$

These two equations can also be written in the form

$$(\mathbf{t}_1\times\mathbf{t}_2)\cdot\mathbf{R} - (\mathbf{R}\cdot\mathbf{t}_2)(\delta\omega\cdot\mathbf{R}) + (\mathbf{t}_2\cdot\delta\omega)(\mathbf{R}\cdot\mathbf{R}) = 0 \quad (4)$$
$$(\mathbf{t}_1\times\mathbf{t}_2)\cdot\mathbf{R} - (\mathbf{R}\cdot\mathbf{t}_1)(\delta\omega\cdot\mathbf{R}) + (\mathbf{t}_1\cdot\delta\omega)(\mathbf{R}\cdot\mathbf{R}) = 0 \quad (5)$$

As these equations are quadratic in $\mathbf{R}$, we see that the surfaces in view, which can give rise to ambiguous motions, must be quadratic.

Now let us find out what kind of quadric they are. Are they ellipsoids, hyperboloids of one or two sheets, or are they of degenerate form?

If we substitute $\mathbf{R} = K\mathbf{t}_2$ into (4), the equation holds. We conclude that a line parallel to $\mathbf{t}_2$ passing through the origin lies entirely in the surface, and thus the surface must be a hyperboloid of one sheet (or one of its degenerate forms).

A hyperboloid of one sheet has two sets of intersecting rulings. It can be verified that $\mathbf{R} = K\mathbf{t}_0$, where $\mathbf{t}_0 = ((\mathbf{t}_2\times\mathbf{t}_1)\times\delta\omega)\times(\mathbf{t}_2\times\mathbf{t}_1)$ is the equation of a second line embedded in the surface.

(b) Since there is no constant term in (4) and (5) the hyperboloids described by these equations pass through the origin, that is, the viewer must be on the surface being viewed. Thus we find that the motion field corresponds in some image regions to points on the surface lying in front of the viewer and in other image regions to points on the surface behind the viewer. A real ambiguity can only arise if the field of view is restricted to regions where both $Z_1 > 0$ and $Z_2 > 0$.

2. (a) If we substitute into equation $Z = Z_0 + pX + qY$ the image coordinates, that is, $x = \frac{X}{Z}$ and $y = \frac{Y}{Z}$ we obtain

$$\frac{Z_0}{Z} = 1 - px - qy$$

(b) The equations for the motion field are

$$\begin{aligned} u &= \frac{1}{Z}(-U + xW) + \alpha xy - \beta\left(x^2+1\right) + \gamma y \\ v &= \frac{1}{Z}(-V + yW) + \alpha\left(y^2+1\right) - \beta xy - \gamma x \end{aligned}$$

Substituting from (a) for $\frac{1}{Z}$ we obtain

$$\begin{aligned} u &= \frac{1}{Z_0}(-U + xW)(1 - px - qy) + \alpha xy - \beta\left(x^2+1\right) + \gamma y \\ v &= \frac{1}{Z_0}(-V + yW)(1 - px - qy) + \alpha\left(y^2+1\right) - \beta xy - \gamma x \end{aligned}$$

(c) For two sets of rigid motions and corresponding surfaces which give rise to the same motion field, we have

$$\begin{aligned} &\frac{1}{Z_{0_1}}(-U_1 + xW_1)(1 - p_1x - q_1y) + \alpha_1 xy - \beta_1\left(x^2+1\right) + \gamma_1 y \\ &\quad = \frac{1}{Z_{0_2}}(-U_2 + xW_2)(1 - p_2x - q_2y) + \alpha_2 xy - \beta_2\left(x^2+1\right) - \gamma_2 y \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{Z_{0_1}}(-V_1 + yW_1)(1 - p_1x - q_1y) + \alpha_1\left(y^2+1\right) - \beta_1 xy + \gamma_1 x \\ &\quad = \frac{1}{Z_{0_2}}(-V_2 + yW_2)(1 - p_2x - q_2y) + \alpha_2\left(y^2+1\right) - \beta_2 xy - \gamma_2 x \end{aligned}$$

We substitute

$$\begin{array}{lll} U_1' = U_1/Z_{0_1} & U_2' = U_2/Z_{0_2} & V_1' = V_1/Z_{0_1} \\ V_2' = V_2/Z_{0_2} & W_1' = W_1/Z_{0_1} & W_2' = W_2/Z_{0_2} \end{array}$$

and equate the coefficients of the powers of x and y that occur in the above equations to derive the following eight conditions

$$\begin{aligned} V_1' - V_2' &= \alpha_1 - \alpha_2 \\ W_1'q_1 - W_2'q_2 &= \alpha_1 - \alpha_2 \\ U_1' - U_2' &= \beta_2 - \beta_1 \\ W_1'p_1 - W_2'p_2 &= \beta_2 - \beta_1 \\ U_1'q_1 - U_2'q_2 &= \gamma_2 - \gamma_1 \\ V_1'p_1 - V_2'p_2 &= \gamma_1 - \gamma_2 \\ U_1'p_1 - U_2'p_2 &= W_2' - W_1' \\ V_1'q_1 - V_2'q_2 &= W_2' - W_1' \end{aligned}$$