Final exam: ***CMSC/AMSC460*** ***Computational Methods*** ***12/19/2005***

120 minutes. **with Partial solutions**

1. Least squares fitting:

a. Write down the matrix A and the right hand side vector b that will allow the coefficient vector $[c_1, c_2, c_3]^t$ of the least-squares fit parabola

$$y = c_1+c_2 x+c_3 x^2$$

to be found via the MATLAB expression c=A\b. (8 pts)

If the data are given as $[x_i,y_i]$, $i=1,\ldots,N$, then the matrix can be written as $A = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ \vdots & \vdots & \vdots \\ 1 & x_1 & x_N^2 \end{bmatrix}, \qquad b = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$

b. Solve the system of equations via the normal equation method and find the coefficients $[c_1, c_2, c_3]^t$ that fit the data points (−3, 1), (−1,−1), (1, 1), (3,−1). (7 pts)

$$A = \begin{bmatrix} 1 & -3 & 9 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

In the normal equation method we solve the equation $A^t Ac = A^t b$

$$A^t A = \begin{bmatrix} 4 & 0 & 20 \\ 0 & 20 & 0 \\ 20 & 0 & 164 \end{bmatrix}, \qquad A^t b = \begin{bmatrix} 0 \\ -4 \\ 0 \end{bmatrix}$$

The middle equation gives us $20\ c_2=-4$, or $c_2=-0.2$. The other two equations are trivial, and give us $c_1=c_2=0$.

So the least squares parabola fit is $y=-0.2\ x$

2. Write down true/false or brief answers

   a. Gaussian integration uses equally spaced points (T/F) (2 pts)

False

   b. Solving a least-squares problem via the normal equations rather than the QR decomposition leads to a worse conditioned algorithm (T/F) (2 pts)

   True

   c. The cost of computing the Fourier transform using Matlab's FFT routine for a vector of length n is $O(n \log_2 n)$ (T/F)

   True (2 pts)

   d. The MATLAB function ode45 is to be preferred to the function ode23s for solving stiff systems of ordinary differential equations. (T/F) (2 pts)

   False

e. What is the main advantage of the Adams-Bashforth methods over the Runge-Kutta methods of the same order, for solving an ordinary differential equation? (4 pts)

After some initial steps they require only one function evaluation per step, where as the Runge-Kutta methods need as many function evaluations as the order of the method.

f. The power iteration method allows us to determine all the eigenvalues and eigenvectors of all matrices. (T/F) (2 pts)

False

g. Find an orthogonal matrix Q and a number z so that $Q\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}z\\0\end{bmatrix}$. (6 pts)

There are many ways to get this. $\begin{bmatrix}0.6 & 0.8\\-0.8 & 0.6\end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}5\\0\end{bmatrix}$

h. Write down how you would solve a rank-deficient least-squares problem using the singular value decomposition. (10 pts)

3. Write down a commented Matlab function

```
[intval, err, ncalls] = MyAdaptQuad(fname,a,b,tol)
```

that performs adaptive quadrature using the Simpson rule, and subdividing the domain of integration to estimate $I(f)=\int_a^b f(x)dx$ for a given function *f(x),* that is programmed in the function **fname**.. The function should estimate the error and stop when its estimate is below **tol.** Include appropriate comments. The function should return the integral value (**intval**), the error estimate (**err**), and the number of calls to the function (**ncalls**). Your work will be assessed for correctness of the algorithms, and explanations in the comments. Minor syntax errors are fine. (25 pts)

4. Let $dy/dt = y^2 - 5t,$ with $y(0) = 1$

Using a step size of $h=0.1$, evaluate $y(0.2)$

a) Use the forward Euler Method, (8 pts)

$y(0)=1$ $dy/dt\ (0) = 1^2-5\times0=1$

$y(0.1)=y(0)+h(dy/dt)=1+0.1*1=1.1$ $dy/dt(0.1)=1.1^2-5\times0.1=1.21-0.5=0.71$

$y(0.2)=1.1+0.71*0.1=1.1+0.071=1.171$

b) For each step, use the forward Euler method as a predictor, followed by a correction step using the Backward Euler method. (12 pts)

dy/dt at (0.10) based on euler prediction $= 1.1^2 -0.5=0.71$

$y(0.1)=y(0)+0.5\times(dy/dt\ (0) + dy/dt(0.1))h=1+0.5\times (1+0.71) \times0.1= 1+0.5\times0.171= 1.0855$

$dy/dt(0.1)=1.0855^2-5\times0.1=0.6783$

Prediction for $y(0.2)= y(0.1)+0.1\times0.6783=1.0855+0.06783=1.15333$

Derivative at $0.2=1.15333^2-1=0.15333\times2.\ 15333=0.3302$

$y(0.2)=y(0.1)+0.5\times(dy/dt\ (0.1)+ dy/dt(0.2))\ h = 1.0855 + 0.05 \times (0.6783 + 0.3302) = 1.0855 + 0.5 \times 0.10085 = 1.1359$

c) Evaluate the stability of the system at $t=0$. (10 pts)

To explore the stability we look at the solution of the linearized equation

$dy/dt=2y$ $y=e^{2t}$

This is a growing solution and the solution may be unstable at t-0.