

Computing the factorization

- QR is useful ... so how do we factorize a matrix A?
- In LU we computed an upper triangular matrix by computing and adding multiples of other rows so that elements below a given column were zeroed out
- The multipliers were stored in L which gave us $A=LU$
- Here we want to zero out entries below the diagonal but do it with orthogonal matrices
- Two strategies
- Zero out a column at a time using a matrix Q_1 so that $Q_1^t A$ gives us all entries below a certain one in a column as zero
 - Householder transformations
 - Result $Q_n^t \dots Q_2^t Q_1^t A = R$ or $A = Q_1 \dots Q_{n-1} Q_n R = QR$
- Zero out one specific entry of a column at a time
 - Givens rotations
- Product of orthogonal matrices is orthogonal

To compute QR

- Perform a sequence of orthogonal transformations that zero out elements
- Orthogonal transformations can be rotations or reflections or combinations
- Givens Rotation:
- Givens matrix has elements
- $c^2 + s^2 = 1$
- How do we use a rotation to zero out an element?

$$G = \begin{bmatrix} c & s \\ s & -c \end{bmatrix}$$

- Let $z = [z_1 \ z_2]^t$
- We want $Gz = \begin{bmatrix} cz_1 + sz_2 \\ sz_1 - cz_2 \end{bmatrix} = xe_1$
- Eliminate z_2 $(c^2 + s^2)z_1 = cx$, $c = z_1/x$.
- Similarly we get $s = z_2/x$, and $z_1^2 + z_2^2 = x^2$

Givens QR

- To apply idea to larger matrix, embed the Givens matrix in identity matrix. We will use the notation G_{ij} to denote an $n \times n$ identity matrix with its i th and j th rows modified to include the Givens rotation: for example, if $n = 6$, then

$$G_{25} = \begin{bmatrix} \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{c} & 0 & 0 & \mathbf{s} & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & \mathbf{s} & 0 & 0 & \mathbf{-c} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{bmatrix},$$

and multiplication of a vector by this matrix leaves all but rows 2 and 5 of the vector unchanged.

- Algorithm for $i=1, \dots, n$

for $j=i+1, \dots, m$

Find Givens matrix G_{ij} to zero out j, i element of A using the the value at position (i, i)

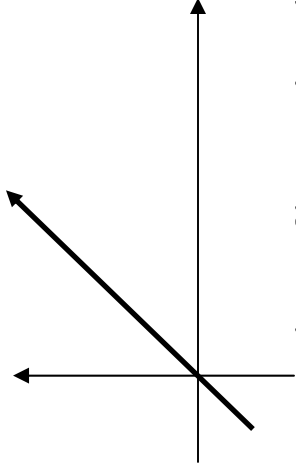
$$A = G_{ij} A$$

end

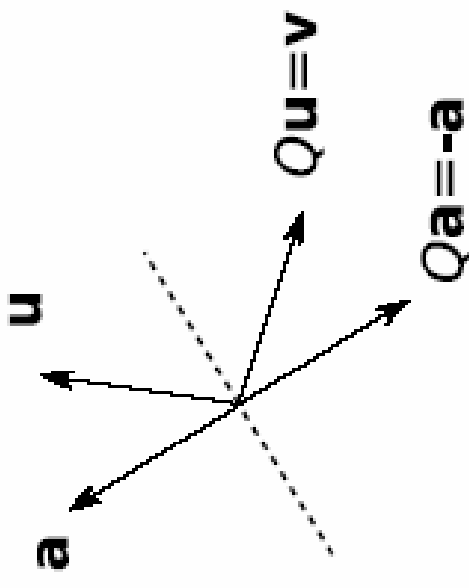
end

Goal: reflect axial vector through a (hyper)-plane

- Axial vector is any vector through the origin. Let it be \mathbf{u}



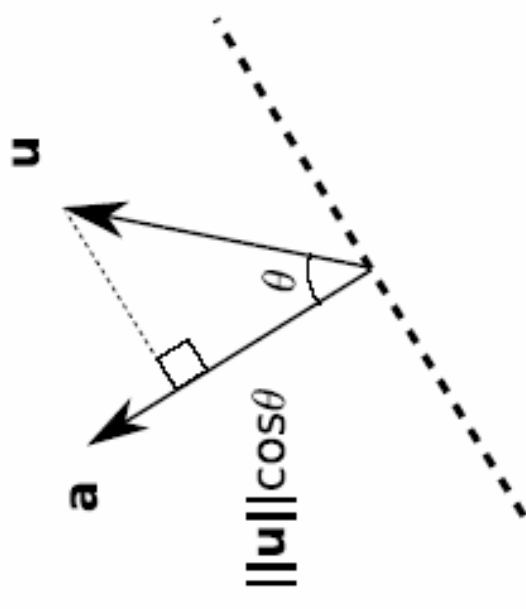
- Goal reflect it through a plane



- Let \mathbf{a} be the unit vector normal to the plane

- $\mathbf{u} \cdot \mathbf{a} = \|\mathbf{u}\| \cos \theta$

- To get reflected vector in the plane
Subtract twice the component of \mathbf{u}
along \mathbf{a}



- $\mathbf{v} = \mathbf{u} - 2 \mathbf{a} \|\mathbf{u}\| \cos \theta$

- $\cos \theta = (\mathbf{a} \cdot \mathbf{u}) / (\|\mathbf{u}\|)$

Householder transform

- Achieve this reflection via multiplication by an orthogonal matrix

- $\mathbf{v} = \mathbf{Q}\mathbf{u} = \mathbf{u} - 2 \frac{\mathbf{a}\|\mathbf{u}\|}{\|\mathbf{a}\|} \cos \theta$

$$\square = \mathbf{u} - 2 \frac{\mathbf{a}\|\mathbf{u}\|}{\|\mathbf{a}\|} (\mathbf{a} \cdot \mathbf{u}) / (\|\mathbf{u}\|)$$

- $= (\mathbf{I} - 2 \frac{\mathbf{a}\mathbf{a}^t}{\|\mathbf{a}\|^2})\mathbf{u}$

- What if \mathbf{a} is not a unit vector?

$$(\mathbf{I} - 2 \frac{\mathbf{a}\mathbf{a}^t}{\mathbf{a}^t \mathbf{a}})\mathbf{u}$$

- $\mathbf{Q} = \mathbf{I} - 2 \frac{\mathbf{a}\mathbf{a}^t}{\mathbf{a}^t \mathbf{a}}$

Householder Transformations

The *Householder transformation* determined by vector v is:

$$H = I - 2 \frac{vv^T}{v^T v}$$

vv^T ← outer product, n×n matrix
 $v^T v$ ← inner product, scalar

To apply it to a vector x , compute:

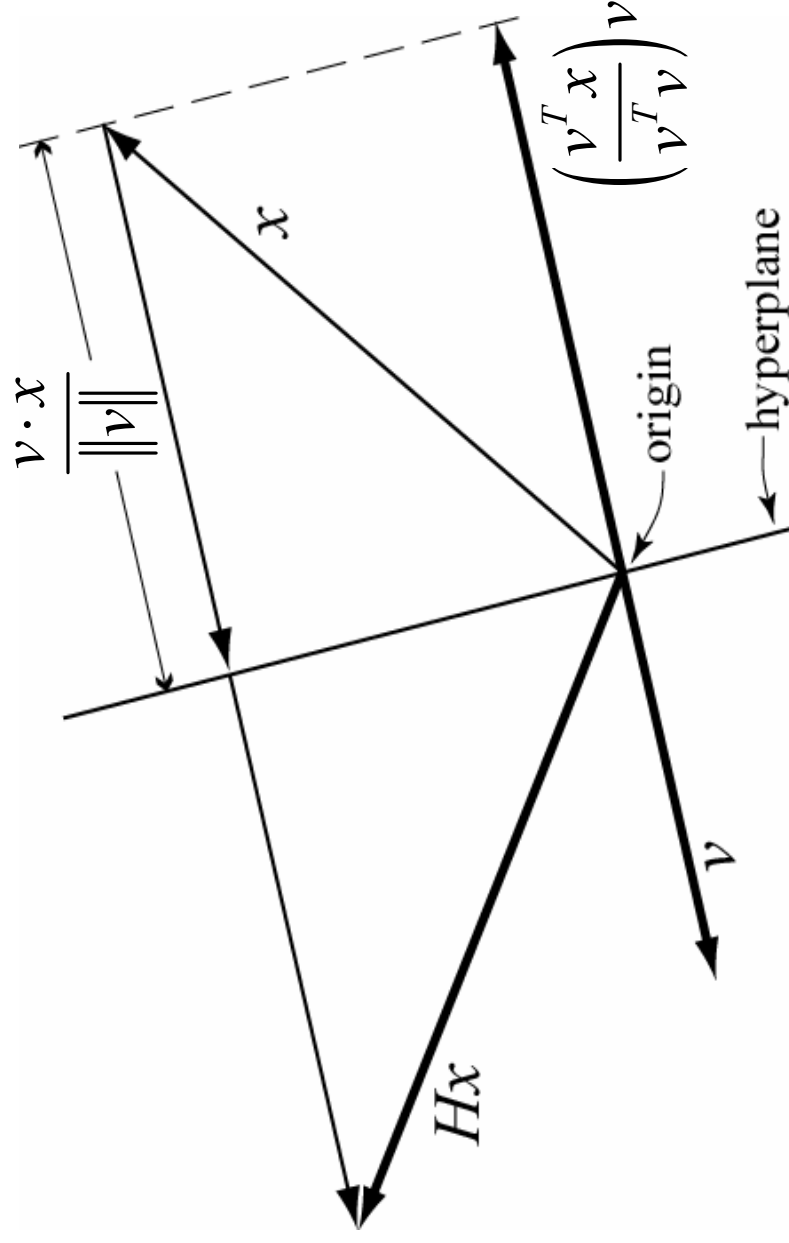
$$Hx = \left(I - 2 \frac{vv^T}{v^T v} \right) x = x - 2 \frac{v(v^T x)}{v^T v}$$

$$Hx = x - \left(2 \frac{v^T x}{v^T v} \right) v$$

← scalar

Householder Geometry

- Hx is x reflected through the hyperplane perpendicular to v ($p : p^T v = 0$)



Householder Properties

- H is symmetric, since

$$H^T = \left(I - 2 \frac{vw^T}{v^T v} \right)^T = I^T - 2 \frac{(vw^T)^T}{v^T v} = I - 2 \frac{v^{TT} v^T}{v^T v} = H$$

- H is orthogonal, since

$$\begin{aligned} H^T H &= HH = \left(I - 2 \frac{vw^T}{v^T v} \right) \left(I - 2 \frac{vw^T}{v^T v} \right) \\ &= I - 4 \frac{vw^T}{v^T v} + 4 \frac{v(v^T v)v^T}{(v^T v)^2} = I - 4 \frac{vw^T}{v^T v} + 4 \frac{vw^T}{v^T v} = I \end{aligned}$$

and $H^T H = I$ implies $H^T = H^{-1}$

Householder to Zero Matrix Elements

We'll use Householder transformations to zero subdiagonal elements of a matrix.

Given any vector a , find the v that determines an H such that,

$$Ha = \alpha e_1 = \alpha [1, 0, 0, \dots, 0]^T$$

Now solve for v :

$$Ha = a - \begin{pmatrix} v^T a \\ 2 \frac{v^T a}{v^T v} v \end{pmatrix} v = a - \mu v = \alpha e_1$$

where μ is parenthesized scalar, related to length of v
 $\Rightarrow v = (a - \alpha e_1) / \mu$

We're free to choose $\mu = 1$, since $\|v\|$ does not affect H

Choosing the Vector v

So $v = a - \alpha e_1$ for some scalar α .

But $\|Ha\|_2 = \|a\|_2$

(prove this by expanding $\|Ha\|_2^2 = (Ha)^T Ha$)

and $\|Ha\|_2 = |\alpha|$ by design, so $\alpha = \pm \|a\|_2$

(either sign will work).

To avoid $v \approx 0$ we choose $\alpha = -\text{sign}(a_1)\|a\|_2$,

so $v = a + \text{sign}(a_1)\|a\|_2 e_1$ is our answer.

Applying Householder Transforms

- Don't compute Hx explicitly, that costs $3n^2$ flops.
- Instead use the formula given previously,

$$Hx = x - \left(2 \frac{v^T x}{v^T v} \right) v$$

- which costs $4n$ flops (if you pre-compute $v^T v$ or pre-normalize $v^T v=2$).
- Typically, when using Householder transformations, you never compute the matrix H ; it's only used in derivation and analysis.

QR Decomposition

- Householder transformations are a good way to zero out subdiagonal elements of a matrix.

- A is decomposed:

$$Q^T A = \begin{bmatrix} R \\ \mathbf{0} \end{bmatrix} \quad \text{or} \quad QQ^T A = A = Q \begin{bmatrix} R \\ \mathbf{0} \end{bmatrix}$$

- where $Q^T = H_n \dots H_2 H_1$ is the orthogonal product of Householders and R is upper triangular.
- Overdetermined system $Ax=b$ is transformed into the easy-to-solve

$$\begin{bmatrix} R \\ \mathbf{0} \end{bmatrix} x = Q^T b$$