AMSC 460/CMSC 460 Final Exam 120 minutes Spring 2007

1_Write down true or false. (3 points each)

a) The formula $y_{n+1}=y_n+0.5*h*(f_{n+1}+f_n)$ is an implicit formula for solving ordinary differential equations T

b) The package ode23s performs worse in integrating a stiff ordinary differential equation than ode45 F

c) Adaptive numerical integration routines use fewer points where the function is changing rapidly F

d) The FFT allows the calculation of the Fourier transform of a N vector for a cost of $O(N^2)$ operations. F

2. Given the system of ordinary differential equations

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2u - 2\cos v \\ u + tv^2 \end{bmatrix}$$

and the initial condition $u(0)=1$ $v(0)=\pi$

a) write down the symbolic expression for the Jacobian matrix that determines stability for any value of t 10 points

b) Write down the numerical value of this matrix using the values given for $t=0$. *4 points*

c) Determine the eigenvalues and eigenvectors of this matrix 10 points

$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} 2u - 2\cos v \\ u + tv^2 \end{bmatrix}$, So the Jacobian is $\begin{bmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{bmatrix} = \begin{bmatrix} 2 & 2\sin v \\ 1 & 2tv \end{bmatrix}$

Substituting the initial condition values $(u = 1, t = 0, v = \pi / 2)$, the Jacobian matrix is $\begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}$.

This has eigenvalues given by the solution of

$-\lambda(2-\lambda) - 2 = 0$ or $\lambda^2 - 2\lambda - 2 = 0$ or $\lambda = \left(2 \pm \sqrt{4+8}\right)/2$ or $\lambda = \left(1 \pm \sqrt{3}\right)$

Inserting the roots we find that the matrix respectively becomes

$$\begin{bmatrix} 1-\sqrt{3} & 2 \\ 1 & -1-\sqrt{3} \end{bmatrix} \qquad \begin{bmatrix} 1+\sqrt{3} & 2 \\ 1 & -1+\sqrt{3} \end{bmatrix}$$

Multiplying the second row in the first case by $1-\sqrt{3}$ and in the second case by $1+\sqrt{3}$ we get

$\begin{bmatrix} 1-\sqrt{3} & 2 \\ 1-\sqrt{3} & \left(-1-\sqrt{3}\right)\left(1-\sqrt{3}\right) \end{bmatrix}$ and $\begin{bmatrix} 1+\sqrt{3} & 2 \\ 1+\sqrt{3} & \left(-1+\sqrt{3}\right)\left(1+\sqrt{3}\right) \end{bmatrix}$, or $\begin{bmatrix} 1-\sqrt{3} & 2 \\ 1-\sqrt{3} & 2 \end{bmatrix}$ and $\begin{bmatrix} 1+\sqrt{3} & 2 \\ 1+\sqrt{3} & 2 \end{bmatrix}$

As expected the eigenvalues make the matrix degenerate, and the eigenvector is a vector orthogonal to the degenerate vector

$\begin{bmatrix} -2 \\ 1-\sqrt{3} \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1+\sqrt{3} \end{bmatrix}$ can be taken as the corresponding eigen vectors

d) Are solutions of this ode stable at *t=0?* *5 points*

Since one of the eigenvalues is positive we will have a growing solution and so the ode is not stable at *t=0*

3) Suppose we have 25 data points (x_i, y_i) with $x_1 < x_2 < \ldots < x_n$.

Write down a formula that approximates

$$Q = \int_{x_1}^{x_n} y(x)\,dx$$

using the values of (x_i, y_i) given. 20 points

Any formula is acceptable. The trapezoidal rule would have been

$$Q = 0.5(x_2 - x_1)(y_1 + y_2) + 0.5(x_3 - x_2)(y_2 + y_3) + \cdots + 0.5(x_{25} - x_{24})(y_{25} + y_{24})$$

In fact since I did not say these points are equidistant, the other rules would have been a bit more complex to write

What is the order of approximation of the formula you have written? 5 points

As discussed in class the error in the composite trapezoid rule is proportional to the domain size (which is $[x_{25}-x_1]$, the local interval squared (h^2), and the second derivative. The trapezoidal rule is a first order method (or another acceptable way to say it would have been that it is exact for linear functions)

4) Provide brief answers

a. Given a a $m \times n$ matrix A. Write down the following decompositions. In each case mention the restrictions on m, n; the types of matrices that form the decomposition (orthogonal, diagonal, triangular), the dimensions of the resulting matrices, and where such a decomposition is used in applications (8 points each)

i. QR decomposition

ii. Singular Value Decomposition

iii. Eigenvalue decomposition

b. Write down the QR algorithm for computing eigenvalues and eigenvectors of a matrix 10 points

$A_0 = A$

for k=1,2,…

$Q_k R_k = A_k$

$A_{k+1} = R_k Q_k$

end

A_{k+1} tends to an upper triangular matrix with the same eigenvalues as A. These eigenvalues lie along the main diagonal of A_{k+1}. Once the eigenvalues are known, the eigenvector can be solved by solving the homogeneous linear systems.