Computational Methods
CMSC/AMSC/MAPL 460

Partial Differential Equations

Ramani Duraiswami,
Dept. of Computer Science

Partial Differential equations

- Given a differential equation involving a function of a single variable we get an ordinary differential equation
- Partial differential equations involve the solution of functions of more than one variable
- Typically involve "boundary conditions" on spatial domains and "initial conditions" on time
- Several methods to numerically solve them
- Here we will look at the simplest
  - Finite difference methods on "regular regions"

Initial and Boundary value problems

- Initial value problems are the kind we have seen so far in the ODE part of the course
- Involve finding y(t), given an ordinary differential equation and initial conditions $y(t_0)$
- We may also can have boundary value problems
- Given a differential equation and conditions at two values of the independent variable
- $y(t_0)=y_0$
- $y(t_1)=y_1$

Laplace's Equation and Poisson's Equation

- Laplacian

In one space dimension

$$\triangle = \frac{\partial^2}{\partial x^2}$$

In two space dimensions

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

- Poisson's equation

$$\triangle u = f(\vec{x})$$

- Laplace's equation
- *f=0*
- *"Elliptical Equations"*

Heat equation

Heat equation

$$\frac{\partial u}{\partial t} = \triangle u - f(\vec{x})$$

$$u(\vec{x}, 0) = u_0(\vec{x})$$

- Parabolic equation
- Boundary conditions as for Laplace, plus initial conditions at time *t=0*

Wave equation

$$\frac{\partial^2 u}{\partial t^2} = \triangle u$$

$$u(\vec{x}, 0) = u_0(\vec{x})$$

$$\frac{\partial u}{\partial t}(\vec{x}, 0) = 0$$

- Hyperbolic equation
- Boundary conditions on space, plus two initial conditions

Finite difference methods

- Approximate the action of operator on function values

In one dimension

$$a \le x \le b$$

$$h = (b-a)/(m+1)$$

$$x_i = a + ih,\ i = 0, \ldots, m+1$$

$$\triangle_h u(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$$

Two dimensions

$$(x_i, y_j) = (ih, jh)$$

$$\triangle_h u(x,y) = \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2} + \frac{u(x,y+h) - 2u(x,y) + u(x,y-h)}{h^2}$$

Laplacian Stencil

$$
\begin{aligned}
P &= (x, y) \\
N &= (x, y + h) \\
E &= (x + h, y) \\
S &= (x, y - h) \\
W &= (x - h, y)
\end{aligned}
$$

$$\triangle_h u(P) = \frac{u(N) + u(W) + u(E) + u(S) - 4u(P)}{h^2}$$