Constrained Optimization and Lagrange Multipliers

Introduction

- In addition to the objective function, we have constraints
  - Point that minimizes objective function may not satisfy constraints
  - Among all points that satisfy constraints (are "feasible") find the best one
  - Alternately, find the one that is locally best
- Solutions may not exist
  - Constraints may be too restrictive

Statement and Notation

$$\min_{\mathbf{x}} f(\mathbf{x})$$

$$\begin{aligned} c_i(\mathbf{x}) &= 0\,, \quad i \in \mathcal{E} \\ c_i(\mathbf{x}) &\geq 0\,, \quad i \in \mathcal{I} \end{aligned}$$

Equality constraints

Inequality constraints

where f and c_i are $\mathcal{C}^2$ functions from $\mathcal{R}^n$ into $\mathcal{R}^1$.

We say that $\mathbf{x}_{opt}$ is a solution to our problem if

- $\mathbf{x}_{opt}$ satisfies all of the constraints.
- For some $\epsilon > 0$, if $\|\mathbf{y} - \mathbf{x}_{opt}\| \leq \epsilon$, and if $\mathbf{y}$ satisfies the constraints, then $f(\mathbf{y}) \geq f(\mathbf{x}_{opt})$.

feasibility

Local optimality

In other words, $\mathbf{x}_{opt}$ is feasible and locally optimal.

Constraints may be active or inactive

Constraints that are active belong to the “active set”

Some examples

- Sometimes constraints eliminate local minima
- Sometimes they add them
- Unconstrained problem: $\min_{x \in \mathrm{R}^n} \|x\|_2^2,$
  - Has unique solution *x*=0
- Constrained problem:

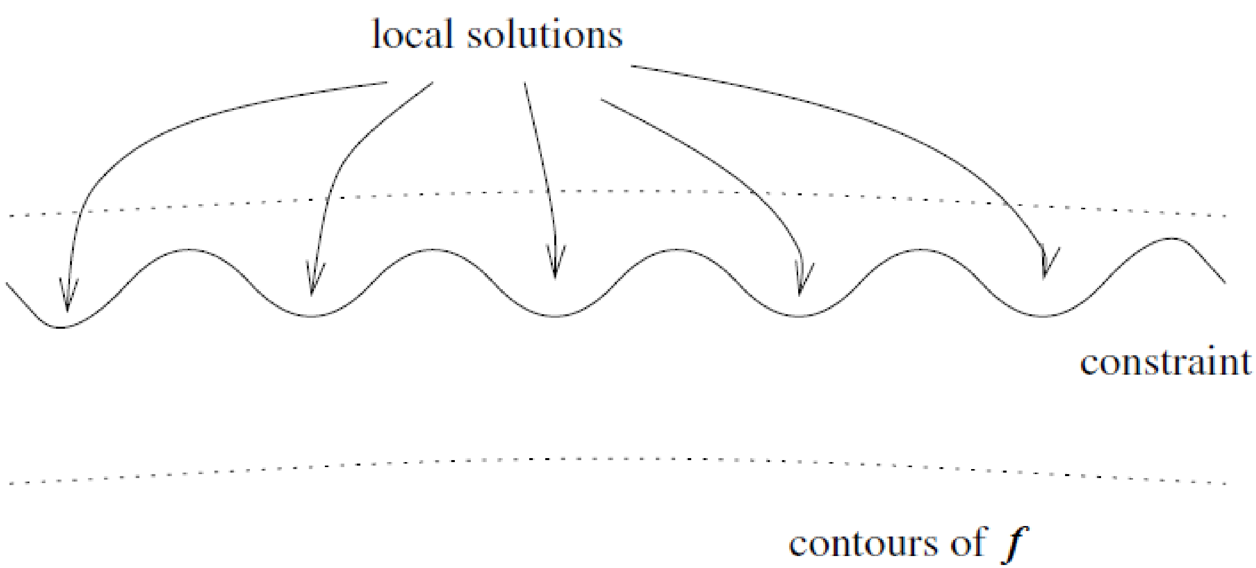
$$\min_{x \in \mathrm{R}^n} \|x\|_2^2, \quad \text{subject to } \|x\|_2^2 \geq 1.$$

- Has infinite solutions --- any point on the unit hypersphere

Constraints may cause many local solutions

$$\min\, (x_2 + 100)^2 + 0.01x_1^2, \qquad \text{subject to } x_2 - \cos x_1 \geq 0,$$

- Unconstrained cost function is a quadratic with a unique extremum
- Constrained version has many local minima



Lagrange Multipliers

- Equality constrained problems
- Suppose problem is
  - min $f(x_1, \ldots, x_n)$ subject to $c(x_1, \ldots, x_n)=0$
- Form Lagrangian
  - $L(x) = f(x) - \lambda c(x)$
  - Set $dL/dx = 0$ (n equations); and equality constraint c provides one equation
  - Solve system to get n+1 unknowns ($x_1^*, \ldots, x_n^*; \lambda^*$)
- LM allow conversion to unconstrained problem
- Value λ^* determines the sensitivity to the constraint

Multiple Constraints

- Suppose we have a classical problem formulation with k equality constraints

minimize $f(x_1, x_2, ..., x_n)$

Subject to $c_1(x_1, x_2, ..., x_n) = 0$

……

$c_k(x_1, x_2, ..., x_n) = 0$

This can be converted in

minimize $L(x, \lambda) = f(x) - \boldsymbol{\lambda}^T \boldsymbol{c}(x)$

where

λ^T is the transpose vector of Lagrangian multpliers and has length k