Utility

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CS 421: Introduction to Artificial Intelligence

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Many slides courtesy of
Dan Klein, Stuart Russell,
or Andrew Moore

Announcements

- P1
  - Due Tuesday (late days until Thursday)
  - Will post solution Friday morning
- P2 up
  - By end of today, you can complete it
  - Feel free to use anything from (y)our P1 or your P2

Hal's Lottery

- You pay $M to enter my lottery
- I put $1 in the pot
- Now, I start flipping fair coins
  - If the coin = heads, I double the pot
  - If the coin = tails, the game ends and you get the pot
- How much would you pay ($M) to enter my lottery?
- Note, $1 = 30 minutes on P2

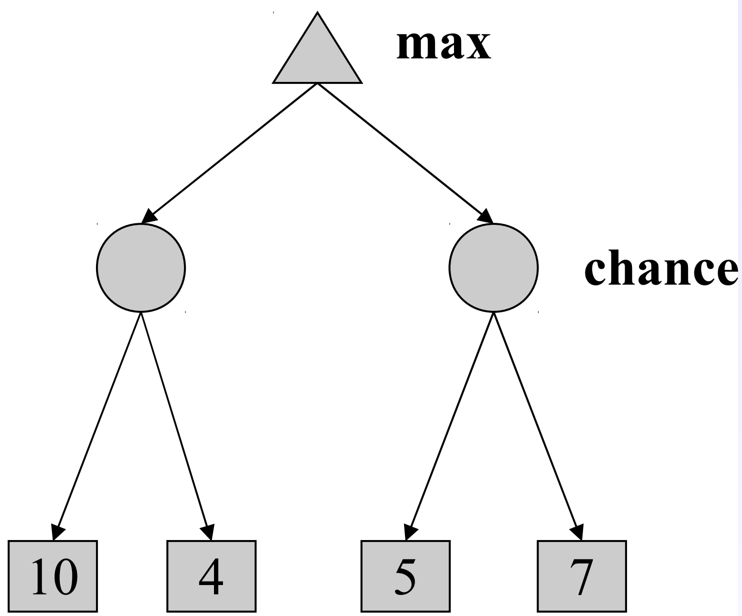
http://u.hal3.name/ic.pl?lottery

Where we are and where we're going

- Where we've been:
  - Single agent, known world, known rewards
  - Multi-agent, known world, known rewards
- Where we're going:
  - Stochastic, known world, known rewards (Markov Decision Processes)
  - Stochastic, ~known world, unknown rewards (Reinforcement Learning)

Expectimax Search Trees

- What if we don't know what the result of an action will be? E.g.,
  - In solitaire, next card is unknown
  - In minesweeper, mine locations
  - In pacman, the ghosts act randomly
- Can do **expectimax search**
  - Chance nodes, like min nodes, except the outcome is uncertain
  - Calculate expected utilities
  - Max nodes as in minimax search
  - Chance nodes take average (expectation) of value of children
- Later, we'll learn how to formalize the underlying problem as a **Markov Decision Process**



Maximum Expected Utility

- Why should we average utilities? Why not minimax?

- Principle of maximum expected utility: an agent should chose the action which maximizes its expected utility, given its knowledge

- General principle for decision making
- Often taken as the definition of rationality
- We'll see this idea over and over in this course!

- Let's decompress this definition…

Reminder: Probabilities

- A random variable is an event whose outcome is unknown
- A probability distribution is an assignment of weights to outcomes

- Example: traffic on freeway?
  - Random variable: T = whether there's traffic
  - Outcomes: T in {none, light, heavy}
  - Distribution: P(T=none) = 0.25, P(T=light) = 0.55, P(T=heavy) = 0.20

- Some laws of probability (more later):
  - Probabilities are always non-negative
  - Probabilities over all possible outcomes sum to one

- As we get more evidence, probabilities may change:
  - P(T=heavy) = 0.20, P(T=heavy | Hour=8am) = 0.60
  - We'll talk about methods for reasoning about probabilities later

What are Probabilities?

- **Objectivist / frequentist answer:**
  - Averages over repeated *experiments*
  - E.g. empirically estimating P(rain) from historical observation
  - Assertion about how future experiments will go (in the limit)
  - New evidence changes the *reference class*
  - Makes one think of *inherently random* events, like rolling dice

- **Subjectivist / Bayesian answer:**
  - Degrees of belief about unobserved variables
  - E.g. an agent’s belief that it’s raining, given the temperature
  - E.g. pacman’s belief that the ghost will turn left, given the state
  - Often *learn* probabilities from past experiences (more later)
  - New evidence *updates beliefs* (more later)

Dutch Books

Horse	Odds	Price
1	Even	$100
2	3 to 1	$50
3	4 to 1	$40
4	9 to 1	$20

- If your internal beliefs don't obey the rules of probability:
  - I can construct a Dutch book
  - ==> I can take infinite amounts of money from you!

Uncertainty Everywhere

- Not just for games of chance!
  - I'm sniffling: am I sick?
  - Email contains "FREE!": is it spam?
  - Tooth hurts: have cavity?
  - 60 min enough to get to the airport?
  - Robot rotated wheel three times, how far did it advance?
  - Safe to cross street? (Look both ways!)
- Why can a random variable have uncertainty?
  - Inherently random process (dice, etc)
  - Insufficient or weak evidence
  - Ignorance of underlying processes
  - Unmodeled variables
  - The world's just noisy!
- Compare to *fuzzy logic*, which has *degrees of truth*, or rather than just *degrees of belief*

Reminder: Expectations

- Often a quantity of interest depends on a random variable

- The expected value of a function is its average output, weighted by a given distribution over inputs

- Example: How late if I leave 60 min before my flight?
  - Lateness is a function of traffic:
    L(none) = -10, L(light) = -5, L(heavy) = 15
  - What is my expected lateness?
    - Need to specify some belief over T to weight the outcomes
    - Say P(T) = {none: 2/5, light: 2/5, heavy: 1/5}
    - The expected lateness:

$$E_{P(T)}[L(T)] = \frac{2}{5} \times (-10) + \frac{2}{5} \times (-5) + \frac{1}{5} \times (15)$$

$$P(none)L(none) + P(light)L(light) + P(heavy)L(heavy)$$

Reminder: Expectations

- Real valued functions of random variables:

$$f : X \rightarrow R$$

- Expectation of a function of a random variable

$$E_{P(X)}[f(X)] = \sum_x f(x) P(x)$$

- Example: Expected value of a fair die roll

X	P	f
1	1/6	1
2	1/6	2
3	1/6	3
4	1/6	4
5	1/6	5
6	1/6	6

$$1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6}$$

$$= 3.5$$

Two Envelopes Problem

- One envelope contains \$100, the other \$200
- Pick an envelope, then I'll let you switch if you want

- Pick an envelope A
- p(A is \$100) = p(A is \$200) = 0.5
- if A is \$100, then other contains \$200
  if A is \$200, then other contains \$100
- So other contains 2*A with p=0.5 and A/2 with p=0.5
- E[money in other] =
- So you should swap...
- and swap...
- and swap...

Utilities

- Utilities are functions from outcomes (states of the world) to real numbers that describe an agent's preferences

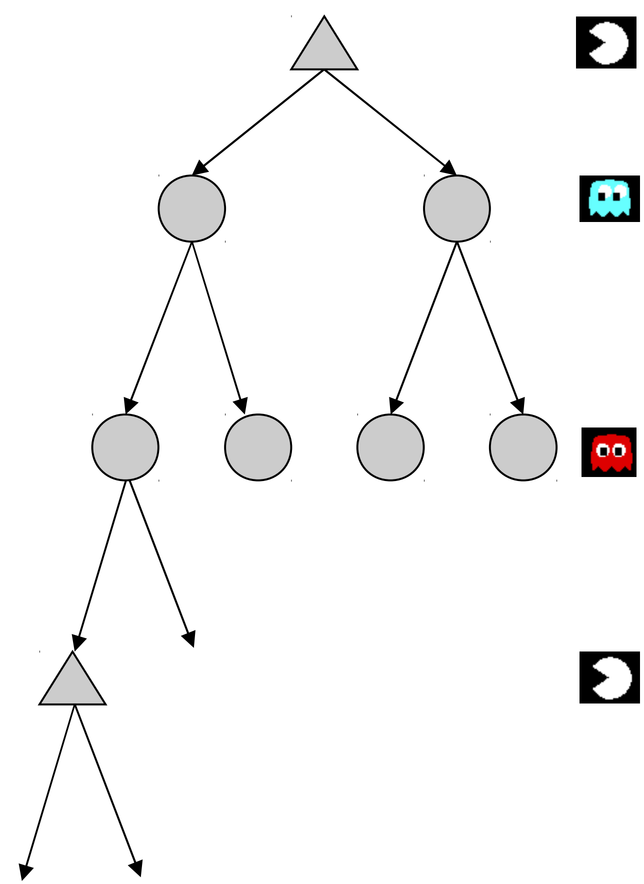
- Where do utilities come from?
  - In a game, may be simple (+1/-1)
  - Utilities summarize the agent's goals
  - Theorem: any set of preferences between outcomes can be summarized as a utility function (provided the preferences meet certain conditions)

- In general, we hard-wire utilities and let actions emerge (why don't we let agents decide their own utilities?)

- More on utilities soon…

Expectimax Search

- In expectimax search, we have a probabilistic model of how the opponent (or environment) will behave in any state
  - Model could be a simple uniform distribution (roll a die)
  - Model could be sophisticated and require a great deal of computation
  - We have a node for every outcome out of our control: opponent or environment
  - The model might say that adversarial actions are likely!
- For now, assume for any state we magically have a distribution to assign probabilities to opponent actions / environment outcomes



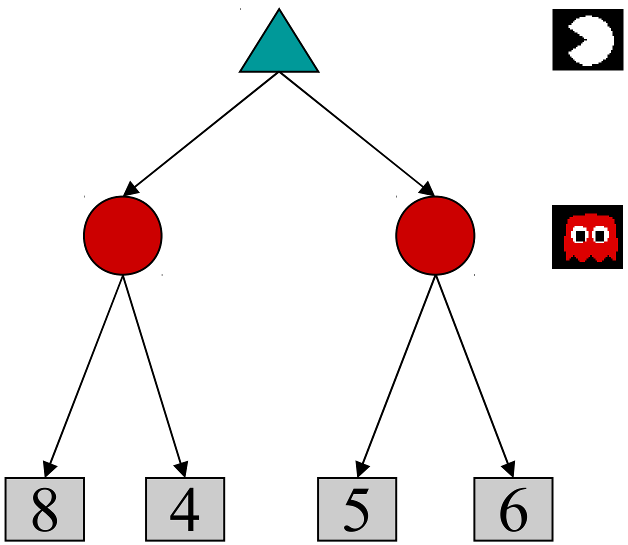
Having a probabilistic belief about an agent's action does not mean that agent is flipping any coins!

Expectimax Pseudocode

```
def value(s)
    if s is a max node return maxValue(s)
    if s is an exp node return expValue(s)
    if s is a terminal node return evaluation(s)

def maxValue(s)
    values = [value(s’) for s’ in successors(s)]
    return max(values)

def expValue(s)
    values = [value(s’) for s’ in successors(s)]
    weights = [probability(s, s’) for s’ in successors(s)]
    return expectation(values, weights)
```



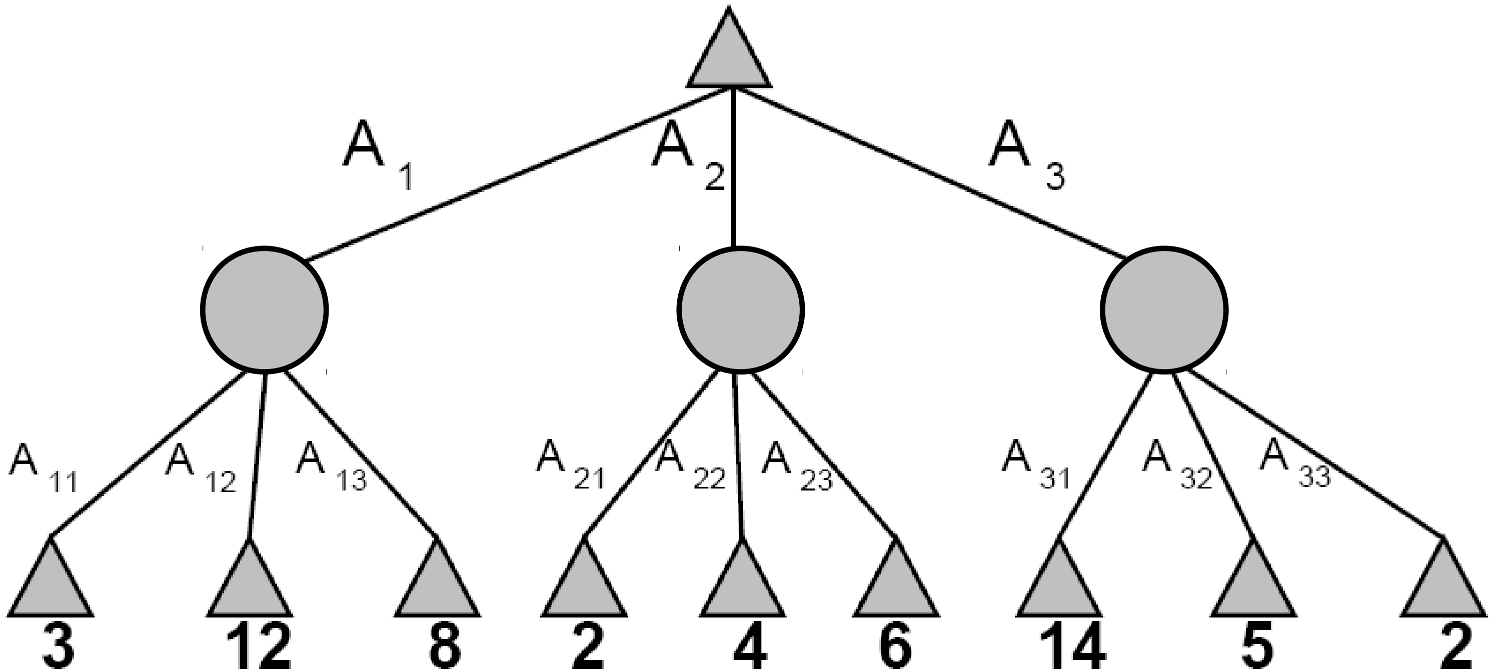
Expectimax for Pacman

- Notice that we've gotten away from thinking that the ghosts are trying to minimize pacman's score
- Instead, they are now a part of the environment
- Pacman has a belief (distribution) over how they will act

- Quiz: Can we see minimax as a special case of expectimax?
- Quiz: what would pacman's computation look like if we assumed that the ghosts were doing 1-ply minimax and taking the result 80% of the time, otherwise moving randomly?

- If you take this further, you end up calculating belief distributions over your opponents' belief distributions over your belief distributions, etc…
  - Can get unmanageable very quickly!

Expectimax Pruning?

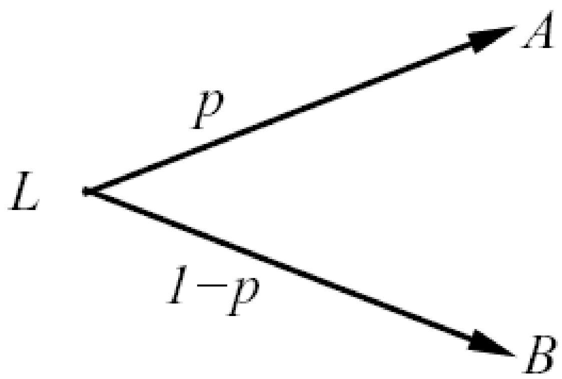


Expectimax Evaluation

- For minimax search, evaluation function insensitive to monotonic transformations
  - We just want better states to have higher evaluations (get the ordering right)
- For expectimax, we need the magnitudes to be meaningful as well
  - E.g. must know whether a 50% / 50% lottery between A and B is better than 100% chance of C
  - 100 or -10 vs 0 is different than 10 or -100 vs 0

Preferences

- An agent chooses among:
  - Prizes: A, B, etc.
  - Lotteries: situations with uncertain prizes



$$L = [p, A;\ (1-p), B]$$

- Notation:

$A \succ B$ $\quad$ A preferred over B

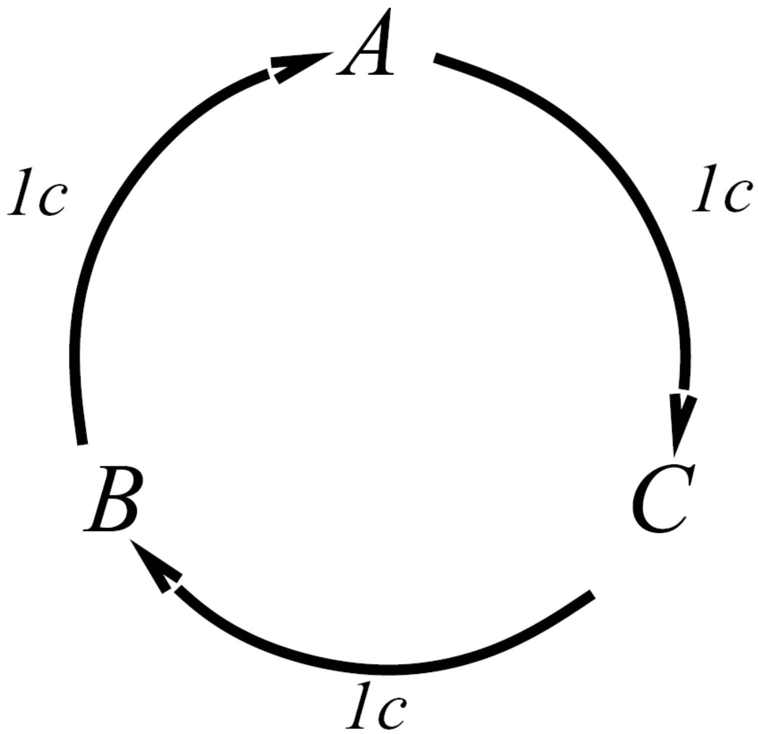
$A \sim B$ $\quad$ indifference between A and B

$A \succeq B$ $\quad$ B not preferred over A

Rational Preferences

- We want some constraints on preferences before we call them rational

- For example: an agent with intransitive preferences can be induced to give away all its money
  - If B > C, then an agent with C would pay (say) 1 cent to get B
  - If A > B, then an agent with B would pay (say) 1 cent to get A
  - If C > A, then an agent with A would pay (say) 1 cent to get C



Rational Preferences

- Preferences of a rational agent must obey constraints.
  - The axioms of rationality:

Orderability

$$(A \succ B) \vee (B \succ A) \vee (A \sim B)$$

Transitivity

$$(A \succ B) \wedge (B \succ C) \Rightarrow (A \succ C)$$

Continuity

$$A \succ B \succ C \Rightarrow \exists p \; [p, A; \; 1-p, C] \sim B$$

Substitutability

$$A \sim B \Rightarrow [p, A; \; 1-p, C] \sim [p, B; 1-p, C]$$

Monotonicity

$$A \succ B \Rightarrow$$
$$(p \geq q \Leftrightarrow [p, A; \; 1-p, B] \succeq [q, A; \; 1-q, B])$$

- Theorem: Rational preferences imply behavior describable as maximization of expected utility

MEU Principle

- Theorem:
  - [Ramsey, 1931; von Neumann & Morgenstern, 1944]
  - Given any preferences satisfying these constraints, there exists a real-valued function U such that:

$$U(A) \geq U(B) \;\Leftrightarrow\; A \succeq B$$

$$U([p_1, S_1;\; \ldots\; ;\; p_n, S_n]) = \sum_i p_i U(S_i)$$

- Maximum expected likelihood (MEU) principle:
  - Choose the action that maximizes expected utility
  - Note: an agent can be entirely rational (consistent with MEU) without ever representing or manipulating utilities and probabilities
  - E.g., a lookup table for perfect tictactoe, reflex vacuum cleaner

Pascal's Wager (d 1662)

- A “proof” that it is a good idea to believe in God

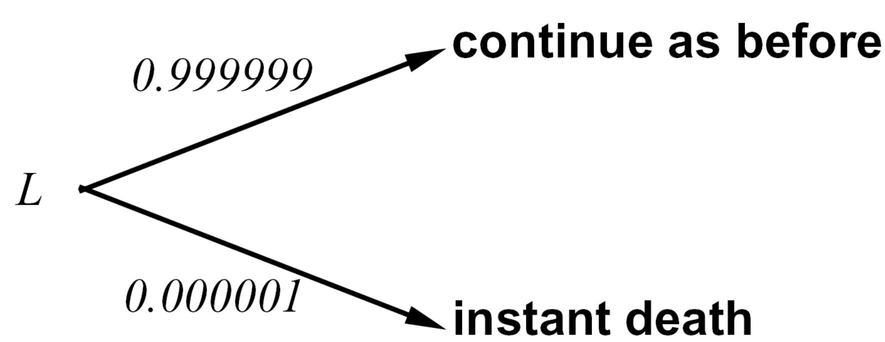
	God Exists	**God Doesn't Exist**
Believe	+infinity	-10
Don't Believe	-infinity	0

- Problems with this argument (mathematically)?
- Problems with this argument (theologically)?
- Exists in many cultures:
  - Islam: al-Juwayni (d 1085)
  - Sanskrit: Sarasamuccaya

Human Utilities

- Utilities map states to real numbers. Which numbers?
- Standard approach to assessment of human utilities:
  - Compare a state A to a standard lottery L_p between
    - ``best possible prize'' u_+ with probability p
    - ``worst possible catastrophe'' u_- with probability 1-p
  - Adjust lottery probability p until A ~ L_p
  - Resulting p is a utility in [0,1]

pay $30 ~



Utility Scales

- Normalized utilities: $u_+ = 1.0$, $u_- = 0.0$
- Micromorts: one-millionth chance of death, useful for paying to reduce product risks, etc.
- QALYs: quality-adjusted life years, useful for medical decisions involving substantial risk
- Note: behavior is invariant under positive linear transformation

$$U'(x) = k_1 U(x) + k_2 \quad \text{where } k_1 > 0$$

- With deterministic prizes only (no lottery choices), only ordinal utility can be determined, i.e., total order on prizes

Example: Insurance

- Consider the lottery [0.5,$1000; 0.5,$0]
  - What is its expected monetary value? ($500)
  - What is its certainty equivalent?
    - Monetary value acceptable in lieu of lottery
    - **http://u.hal3.name/ic.pl?q=insurance**
  - Difference is the insurance premium

Hal's Lottery, revisited

Friendly game	$100	$4.28
Millionaire	$1,000,000	$10.95
Billionaire	$1,000,000,000	$15.93
Bill Gates (2008)	$58,000,000,000	$18.84
US GDP (2007)	$13.8 trillion	$22.79
World GDP (2007)	$54.3 trillion	$23.77
Googolaire	$10^100	$166.50

e pot

- How much would you pay ($M) to enter?

  E[payoff] = (1/2) 1 + (1/4) 2 + (1/8) 4 + (1/16) 8 + ...
  = (1/2) + (1/2) + (1/2) + (1/2) + ...
  = infinity!

- Why weren't we willing to pay $1m to enter?

Money

- Money does not behave as a utility function
- Given a lottery L:
  - Define expected monetary value EMV(L)
  - Usually U(L) < U(EMV(L))
  - I.e., people are risk-averse

- Utility curve: for what probability p

  am I indifferent between:
  - A prize x
  - A lottery [p,$M; (1-p),$0] for large M?
- Typical empirical data, extrapolated with risk-prone behavior:

Example: Human Rationality?

- Famous example of Allais (1953)
  - A: [0.8,$4k; 0.2,$0]
  - B: [1.0,$3k; 0.0,$0]

  - C: [0.2,$4k; 0.8,$0]
  - D: [0.25,$3k; 0.75,$0]

http://u.hal3.name/ic.pl?q=allais

Reinforcement Learning

- [DEMOS]

- Basic idea:
  - Receive feedback in the form of rewards
  - Agent's utility is defined by the reward function
  - Must learn to act so as to maximize expected rewards
  - Change the rewards, change the learned behavior

- Examples:
  - Playing a game, reward at the end for winning / losing
  - Vacuuming a house, reward for each piece of dirt picked up
  - Automated taxi, reward for each passenger delivered

Markov Decision Processes

- An MDP is defined by:
  - A set of states $s \in S$
  - A set of actions $a \in A$
  - A transition function T(s,a,s')
    - Prob that a from s leads to s'
    - i.e., P(s' | s,a)
    - Also called the model
  - A reward function R(s, a, s')
    - Sometimes just R(s) or R(s')
  - A start state (or distribution)
  - Maybe a terminal state

- MDPs are a family of non-deterministic search problems
  - Reinforcement learning: MDPs where we don't know the transition or reward functions