Dynamics of vapor bubbles with nonequilibrium phase transitions in isotropic acoustic fields

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Based on a multiscale asymptotic technique, a weakly nonlinear theory of rectified heat transfer to vapor bubbles is developed. The theory takes into account effects of internal and external heat transfer near the bubbles including memory terms, viscosity and slight compressibility of the liquid, surface tension, and nonequilibrium phase transitions. It is shown that the combination of these effects strongly influences bubble dynamics in high frequency acoustic fields. Results of calculations of bubble dynamics in water and liquid helium are presented. The role of kinetics of phase transitions is analyzed. © 2000 American Institute of Physics. [S1070-6631(00)01601-9]

I. INTRODUCTION

Acoustics of vapor bubbles and acoustic vapor cavitation was intensively studied theoretically and experimentally in the 1960’s and 1970’s. Many of these studies were related to the design of cryogenic bubble chambers for registration of the tracks of charged particles1–3 and measurements of the tensile strength of liquids.4–6 More recent applications include acoustic enhancement of boiling in microgravity7,8 and the use of bubble dynamics for determination of liquid–vapor interface properties.9

Wang10 and Khabeev11 performed a linear analysis of forced vapor bubble oscillations and showed a strong difference in acoustic properties of vapor bubbles and bubbles of noncondensable gas. It is accepted that a gas bubble can be represented as a nonlinear oscillator with the inertia provided by the added mass of the liquid and the restoring force due to the gas elasticity. The resonance frequency of this oscillator is called the Minnaert frequency. The stiffness of vapor bubbles, however, is mainly controlled by evaporation/condensation tending to maintain the vapor pressure close to the saturation pressure. So the nature of the restoring force determining the primary resonance of vapor bubbles is different from that for gas bubbles. In addition to the primary resonance, vapor bubbles exhibit a second resonance in acoustic fields corresponding to smaller sizes or smaller frequencies, which is known as the condensation-evaporation resonance (to differentiate from the primary resonance, it is better to call it condensation-evaporation-capillary resonance, since it is impossible in the absence of phase transitions and capillary effects11). This resonance was first reported by Finch and Neppiras.12 Hsieh,13 Marston,14 and Hao and Prosperetti15 provided physical insight into the second resonance. Prosperetti and co-workers8 noted that the term “resonance” is not correct if applied to the increase of the bubble oscillation amplitude in the low frequency range and the “second resonance” is “a condition of linearly unstable equilibrium.” Despite this we will use the term “resonance” in the present study for simplicity.

The effect of rectified heat transfer in vapor bubbles was investigated theoretically by Akulichev et al.,5,16,17 Tkachev and Shestakov,3,18 Wang,19 Alekseev,20,21 Mironov,22 Patel et al.,23 Gumerov,24 and Hao and Prosperetti.15 It is noteworthy that rectified heat transfer is a substantially nonlinear effect which occurs due to the difference in the amounts of the condensed and evaporated liquid during a period of forced bubble oscillations. This difference can be negative or positive and, consequently, it forces the bubble to grow or collapse on average. At some specific bubble sizes this difference can be zero which results in stable or unstable steady bubble oscillations. For example, Marston and Greene25 observed stable oscillations of bubbles for several seconds in liquid helium-I.

Tkachev and Shestakov,3,18 Akulichev,17 and Hao and Prosperetti15 theoretically studied rectified heat transfer using numerical solutions of the governing equations. Alekseev20,21 used the method of averaging, and Gumerov24 employed a multiscale technique to obtain equations describing the behavior of the mean radius of a vapor bubble under acoustic action. These asymptotic solutions are formally valid for small amplitude oscillations. Fyrillas and Szeri26 developed an asymptotic method to solve the problem of rectified diffusion for thin boundary layers. However, for finite amplitude perturbations the method provides only a quasisteady solution for the smooth (averaged) concentration field. The transient solution of the smooth problem was obtained for small amplitude oscillations.

From a physical point of view it is more difficult to justify simulations of strong forced nonlinear oscillations of vapor bubbles in acoustic fields based on spherically symmetrical models. Due to the parametric resonances between volume and shape modes a stable spherical bubble shape can be realized only at relatively small amplitudes.27,28 The recent paper of Hao and Prosperetti29 provides a theory and comparisons with experiments on shape stability of oscillating air bubbles. Depending on the excitation frequency, and its ratio to shape mode resonance frequencies, the shape sta-
bility threshold for acoustic amplitudes stretches from 0.01 atm to 1 atm and higher (for ambient atmospheric pressures). Shape stability thresholds for vapor bubbles can differ from those for gas bubbles. Other effects, such as acoustic streaming, translatory bubble motion, and gravitation, can also substantially influence the results. The ratio between the bubble size and acoustic wavelength influences the shape of the bubble.

Another simplification present in the vast majority of theories, which substantially limits their applicability, is the assumption of local thermodynamic equilibrium of the vapor/liquid interface, or quasiequilibrium character of evaporation condensation. The argument for neglecting the kinetics of phase transitions is that it should be important only for high mass flux rates corresponding to velocities of the vapor about the speed of sound or for very high frequencies. The frequency range for nonequilibrium kinetics, however, depends substantially on the value of the condensation coefficient, $\beta$, and on temperature. $\beta$ is the fraction of the molecules hitting the liquid which condense. It is also known as the evaporation coefficient and sometimes is used interchangeably with the term “accommodation coefficient.” In the present study we do not differentiate between these terms. At low temperatures and for $\beta$ about 1, which is typical for cryogenic liquids, phase transitions can occur in quasiequilibrium conditions even in the megahertz range. For high temperatures typical for boiling metals and water at atmospheric pressure, the kinetics of evaporation condensation can be important in the kilohertz range.

The real situation is even more complex, since the value of $\beta$ depends strongly on the contamination of the interface and differs for stagnant and mobile interfaces. Even for purified water without any detectable trace contaminants, the range of recently reported values of $\beta$ stretches from 0.006 to 1. Calculations for water vapor bubbles at several values of the accommodation coefficient performed first by Khabeev show that for frequencies of about 20 kHz the linear response functions for bubbles at $\beta=0.1$ and $\beta=1$ are substantially different. The rates of bubble growth due to rectified heat transfer are also appreciably different for $\beta=1$ and $\beta=0.04$ at frequencies of about 10 kHz (with $\beta=0.04$ for water is proposed by Alty and Mackay and supported by Chodes et al. and several other experimenters).

In the present study the earlier model of a vapor bubble with nonequilibrium phase transition is extended to include effects of liquid compressibility, viscosity, and, most importantly, surface tension. The inclusion of surface tension in the model requires modification of the asymptotic multiscale procedure developed in previous studies. From a physical point of view the surface tension plays a significant role in stabilizing forced oscillations of small bubbles. One of the findings of the present study is the possibility of stabilizing small bubbles in high frequency fields near the condensation–evaporation–capillary resonance. For superheated and subcooled liquids there can exist multiple threshold and stable radii of vapor bubbles in acoustic fields. For water boiling at atmospheric pressure, combining the effects of surface tension and nonequilibrium phase transitions shows a strong influence of $\beta$ on the bubble dynamics. This can be utilized for the determination of this coefficient. It was also found that the earlier multiscale procedure does not properly work for approximations of order higher than two. A correct procedure requires introducing slow spatial scales for the heat transfer problem in the liquid. In the present study the method of asymptotic matching was applied for the nonoscillatory part of the temperature field in the liquid and the equation for rectified heat transfer was obtained to the third-order approximation.

II. MODEL

Below we consider a spherically symmetrical model of a one-component vapor bubble in an isotropic pressure field, with the wavelength much larger than the bubble size, $\omega a \ll \pi C$, where $\omega$ is the circular frequency, $a$ is the bubble radius, and $C$ is the speed of sound in the liquid.

For a viscous liquid and inviscid vapor the mass, momentum, and energy conservation equations at the interface can be written in the form:

$$\rho_l \left( \dot{a} - w_{la} \right) = \rho_v \left( \dot{a} - w_{va} \right) = \xi,$$
$$\Pi_{la}^r = -p_v + \xi (w_{va} - w_{la}) + \frac{2}{a} \sigma,$$
$$\Pi_{la}^r w_{la} - q_{la} + \frac{1}{2} \xi w_{la}^2 = -p_v w_{va} - q_{va} + \frac{1}{2} \xi w_{va}^2 + \xi l + \sigma + \frac{2}{a} \sigma \dot{a}.$$

Here $\rho, v, w, q$ are the density, radial velocity, and heat flux, $p$ and $\Pi^r$ are the pressure and radial component of the stress tensor, and $\xi, \sigma, \dot{l}$ and $r$ are the rate of evaporation, surface tension, and latent heat of evaporation. Subscripts $l$ and $v$ refer to liquid and vapor, respectively, and subscript $a$ denotes parameters on the interface.

The dynamic equation describing forced radial bubble oscillation in slightly compressible liquid is well known. However this equation was derived for bubbles of constant mass. Using the Keller and Miksis approach and boundary conditions (1) and (2) we can obtain a modified equation for bubbles of variable mass in the form

$$(1 - \frac{w_{la}}{C}) \dot{w}_{la} + 2 \left[ 1 - \frac{w_{la}}{4C} \right] \dot{w}_{la} - \frac{1}{2} \frac{d}{dt} w_{la}^2 = -\frac{1}{p_l} \left[ 1 + \frac{a}{C} \frac{d}{dt} \right] \left[ \Pi_{la}^r + P_v(t) + \frac{4}{a} \mu_l w_{la} \right],$$

where $\mu_l$ is the liquid viscosity and $P_v(t)$ is the forcing pressure. A similar equation was used recently by Sochard et al. for numerical simulation of gas–vapor bubbles.

Neglecting the temperature jump across the interface, the rate of evaporation can be found from the Hertz–Knudsen–Langmuir kinetic equation:

$$\dot{\xi} = \frac{\beta}{\sqrt{2} \pi R_v T_a} [p_s(T_v) - p_v],$$

where $R_v$ is the gas constant of the vapor, $\beta$ is the accommodation (condensation) coefficient, and $p_s$ is the saturation
pressure, which we assume to be a known function of the interface temperature, $T_a$. To be more specific, the Clausius–Clapeyron equation can be used:

$$\frac{dp_v}{dT} = \frac{1}{T} \left( \frac{1}{\rho_v(T)} - \frac{1}{\rho_l(T)} \right),$$

(6)

where $\rho_v$ is the vapor density on the saturation line.

Assuming that the vapor velocity is much smaller than the speed of sound in the vapor, we can consider the internal bubble pressure, $p_v$, to be spatially uniform:$^{38,42}$

$$p_v = p_v(t).$$

(7)

Modeling the vapor as a perfect gas, of temperature $T_v$, and density $\rho_v$,

$$p_v = \rho_v R_v T_v,$$

(8)

we have the following integral of energy for the vapor:$^{38,43}$

$$a \dot{p}_v + 3 \gamma p_v \dot{w}_{va} + 3(\gamma - 1) \dot{q}_{va} = 0,$$

(9)

where $\gamma$ is the ratio of the vapor specific heats. The vapor radial velocity, temperature, and heat flux at the interface, $\dot{q}_{va}$, can be found using assumptions (7) and (8) and the energy equation:$^{38,42}$

$$w_v = \frac{(\gamma - 1) \lambda_v}{\gamma \rho_v} \frac{\partial T_v}{\partial r} - \frac{r \rho_v}{3 \gamma \rho_v} \frac{\partial T_v}{\partial r} - \rho_v \left( \frac{1}{r^2} \right) \frac{\partial}{\partial r} \left( r^2 \lambda_v \frac{\partial T_v}{\partial r} \right),$$

$$q_{va} = -\lambda_v \left. \frac{\partial T_v}{\partial r} \right|_{r=a}, \quad T_v|_{r=a} = T_a,$$

where $\lambda_v$ is the vapor thermal conductivity, $c_p$ the vapor specific heat, and $r$ is the distance from the bubble center.

The thickness of the thermal boundary layer in the liquid is usually much smaller than the wavelength of sound. Thus, to determine the temperature field in the liquid, $T_l$, we can neglect the effects of liquid compressibility within the thermal boundary layer and use the energy equation in the form

$$\rho_l c_l \left( \frac{\partial T_l}{\partial t} + \frac{a^2 \dot{w}_{la}}{\rho_l} \frac{\partial T_l}{\partial r} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \lambda_l \frac{\partial T_l}{\partial r} \right) + \frac{12 \mu_l \dot{w}_{la}^2 \rho_l}{r^6},$$

with boundary conditions

$$T_l|_{r=a} = T_a, \quad T_l|_{r=a} = T_a, \quad q_{la} = -\lambda_l \left. \frac{\partial T_l}{\partial r} \right|_{r=a}.$$

The above system of equations is closed if the dependencies of the thermophysical coefficients on pressure and temperature are known. In the present study we neglect these dependencies and accept

$$a, \lambda, \beta, \lambda_v, \lambda_l, \mu_l, c_p, c_v, \gamma = \text{const}.$$

We also need to specify function $\dot{p}_v(t)$, which for an acoustic field of amplitude $P_A$ can be written in the form

$$\dot{p}_v(t) = p_v(T_a) + \Delta p + \text{Re}(P_A e^{i\omega t}),$$

(10)

where $\Delta p$ is the difference between the static and saturation pressures. In the present study we limit the analysis to small amplitudes,

$$\frac{P_A}{p_v(T_a)} = \epsilon \ll 1.$$

(11)

At $P_A = 0$ the above equations have a steady state solution, when all fluxes and velocities are zero:

$$a = a_e, \quad w_{la} = 0, \quad T_l = T_e = T_a = T_s,$$

$$p_v = p_s(T_a), \quad \xi = 0, \quad q_{la} = q_{va} = 0.$$

The equilibrium radius, $a_e$, is not arbitrary, and can be determined from (2), (4), and (10):

$$a_e = -\frac{2 \sigma}{\Delta p}.$$

(12)

It is known that this equilibrium is unstable and bubbles with radii $a < a_e$ collapse, while larger bubbles, $a > a_e$. We assume that the acoustic field can stabilize the bubble. If so, then the static pressure perturbations should be considered small:

$$\delta = \frac{1}{\rho_s(T_s)} \left| \frac{\Delta p}{p_s(T_s)} \right| \ll 1, \quad \left| \frac{\Delta p}{p_s(T_s)} \right| \ll 1,$$

(13)

because of small amplitudes of the acoustic field (11) driving the bubble dynamics.

### III. Method of solution

To obtain the equation for rectified heat transfer we use a multiscale asymptotic technique for weakly nonlinear oscillations of drops and bubbles.$^{35,24}$ However, the method must be modified, since the effect of surface tension neglected in Refs. 35 and 24 changes the rank of the system matrix for zero mode of oscillation, and a straightforward application of the technique is impossible.

#### A. Transformation of variables

First, we reduce the number of variables by eliminating $p_v, w_v, p_{la}, w_{la}$, and $\Pi_{la}^{rr}$ from the system. The governing equations can be represented in the following form:

$$\rho_l \dot{a} - \rho_l \dot{w}_{la} - \xi = 0,$$

(14)

$$\left( a + \frac{4 \mu_l}{\rho_l C} w_{la} + \frac{2 \xi}{\rho_l} \right) \dot{w}_{la} + \frac{3}{2} w_{la}^2 + \frac{2 \xi}{\rho_l} \dot{w}_{la}^2 + \frac{w_{la}^2 \xi}{2 C \rho_l} + \frac{a}{\rho_l C} \dot{w}_{la}$$

$$+ \frac{4 \mu_l}{\rho_l a} \dot{w}_{la}$$

$$= \frac{1}{\rho_l C} \left( w_{la}^2 + \frac{\xi}{\rho_l} + \frac{a d}{C} \right) \dot{a}$$

$$\times \left[ p_v - p_v(t) + \left( \frac{1}{\rho_l} \frac{R_v T_a}{\rho_v} \right) \xi \right] - \frac{2 \sigma}{\rho_v a}.$$

(15)

$$a \dot{p}_v + 3 \gamma p_v \dot{a} + 3(\gamma - 1) \dot{q}_{va} - 3 \gamma \xi R_v T_a = 0,$$

(16)
\[
\frac{1}{\rho_l} \left( \frac{p_v - 2\sigma}{\sigma} \right) - R_v T_a - l \right] \xi + q_{va} - q_{la} = \frac{1}{2} \left( \frac{R_v T_a}{p_v} \right)^2 \xi^2.
\]
(17)

\[
\xi = \frac{\beta}{\sqrt{2\pi R_v T_a}} p_v(T_a) - p_v = 0.
\]
(18)

We then perform transform \((r, t) \rightarrow (\eta, t)\) with \(\eta = \rho a(t)\) to fix the moving boundary in the heat transfer problems:

\[
\frac{p_v T_a}{\rho(T_a)\kappa_v} \frac{\partial T}{\partial t} - \frac{\gamma p_v}{\kappa_v} \frac{\partial T}{\partial \eta} = \frac{\gamma p_v}{\kappa_v} \frac{\partial T}{\partial \eta} - \frac{\gamma}{\kappa_v} \frac{\partial T}{\partial \eta}.
\]
(19)

\[
T_{\eta=1} = T_a,
\]

\[
T_{\eta=\infty} = T_\infty,
\]

where \(\kappa_t\) and \(\kappa_v\) are the thermal diffusivities of the liquid and vapor (at ambient conditions).

Next, we introduce dimensionless fast and slow time scales, \(t_0 = \omega t\), and \(t_2 = \omega^2 t\), \(n = 1, 2, \ldots\). All unknowns are considered now as functions of this set of times and the temporal derivatives are represented as series:

\[
\frac{d}{dt} = \omega \left( \frac{\partial}{\partial t_0} + \epsilon \frac{\partial}{\partial t_1} + \epsilon^2 \frac{\partial}{\partial t_2} + \ldots \right).
\]
(20)

Finally, we expand the unknowns in the following asymptotic series:

\[
a(t) = a_0(t_0, t_1, t_2, \ldots) + \left[ 1 + \epsilon a_1(t_0, t_1, t_2, \ldots) + \ldots \right],
\]
(21)

\[
w_{la}(t) = \omega a_{la}(t_0, t_1, t_2, \ldots) + \left[ \epsilon w_{la}(t_0, t_1, t_2, \ldots) + \ldots \right],
\]
(22)

\[
p_v(t) = p_v(T_\infty) \left[ 1 + \epsilon p_v(t_0, t_1, t_2, \ldots) + \ldots \right],
\]
(23)

\[
T_0(t) = T_\infty \left[ 1 + \epsilon T_0(t_0, t_1, t_2, \ldots) + \ldots \right],
\]
(24)

\[
\xi(t) = \omega \rho_v a_0(t_1, t_2, \ldots) \left[ \epsilon \xi(t_0, t_1, t_2, \ldots) + \ldots \right],
\]
(25)

\[
q_{va}(t) = \lambda_v T_\infty \left[ a(t_0, t_1, t_2, \ldots) + \ldots \right] \left[ \epsilon q_{va}(t_0, t_1, t_2, \ldots) + \ldots \right],
\]
(26)

where \(\rho_v\) is the vapor density at \(T_v = T_\infty\) and \(p_v = p_v(T_\infty)\).

### B. Length scales and nondimensional parameters

In the present model we can identify the following independent length scales characterizing different physical phenomena:

\[
L_{T_v} = \left( \frac{\kappa_v}{\omega} \right)^{1/2}, \quad L_{T_l} = \left( \frac{\kappa_l}{\omega} \right)^{1/2},
\]

\[
L_p = \frac{1}{\omega} \left( \frac{p_v(T_\infty)}{\rho_l} \right)^{1/2}, \quad L_\mu = \left( \frac{\mu_l}{\rho_l \omega} \right)^{1/2}, \quad L_\sigma = \frac{\sigma}{p_v(T_\infty)},
\]
(30)

\[
L_C = \frac{C}{\omega}, \quad L_\beta = \frac{\beta l}{(1 - \rho)(2\pi R_v T_\infty)}, \quad L_d = \frac{1}{\omega} \left( \frac{\lambda_v T_\infty}{\mu_l} \right)^{1/2}.
\]

Here \(L_{T_v}\) and \(L_{T_l}\) are the characteristic lengths of temperature penetration into the vapor and liquid, \(L_p\) is the characteristic primary resonance length, \(L_\mu\) is the characteristic thickness of the viscous boundary layer in the liquid, \(L_\sigma\) is the characteristic capillary length, \(L_\beta\) is the inverse wave number, and \(L_d\) are the characteristic lengths connected with the nonequilibrium phase transitions and viscous dissipation in the liquid. The ratios of the bubble radius to these scales produce eight dimensionless parameters. We also introduce the following five dimensionless constants:

\[
k_i = 1 - \frac{1}{\gamma}, \quad k_i = \frac{R_v T_\infty}{l}, \quad \rho = \frac{\rho_v}{\rho_l}, \quad \lambda = \frac{\lambda_v}{\kappa_l},
\]
(31)

All other dimensionless parameters can be represented as a combination of the 14 independent basis parameters—those of (30) and (31) and \(\epsilon\).

### C. Solution of thermal problems

Formally, at \(\epsilon \rightarrow 0\) convective and source terms entering energy equations are small and in the \(m\)th order approximation we have the following inhomogeneous linear equations to determine temperatures outside and inside the bubble, respectively:

\[
\frac{a_0^2}{L_{T_0}^2} \frac{\partial u_m}{\partial t_0} - \frac{1}{\eta^2} \frac{\partial}{\partial \eta} \left( \eta \frac{\partial u_m}{\partial \eta} \right) = f_m,
\]
(32)

\[
\frac{a_0^2}{L_{T_v}^2} \frac{\partial v_m}{\partial t_0} - \frac{k_i a_0^2}{L_{T_v}^2} \frac{\partial p_m}{\partial t_0} - \frac{1}{\eta^2} \frac{\partial}{\partial \eta} \left( \eta \frac{\partial v_m}{\partial \eta} \right) = g_m.
\]
(33)

These are subject to the boundary conditions:

\[
\text{where } f_m(\eta, t_0, t_1, \ldots) \text{ and } g_m(\eta, t_0, t_1, \ldots) \text{ are functions that depend on approximations of order less than } m.
\]

Let us evaluate the range of frequencies, where this scheme is valid. Since for vapor bubbles the heat transfer in
the liquid plays the major role, consider as an example the energy equation for the liquid (19), for which the terms can be evaluated as follows:

\[ \frac{a^2}{\kappa I} \frac{\partial T_I}{\partial t} = \frac{a_0^2 \Delta T}{\kappa I}, \]

\[ \frac{1}{\eta^2} \frac{\partial}{\partial \eta} \left( \eta^2 \frac{\partial T_I}{\partial \eta} \right) = \frac{a_0^2 \Delta T}{\kappa I}, \]

\[ \frac{a}{\kappa I} \left( \eta - \frac{1}{\eta^2} \right) \frac{\partial T_I}{\partial \eta} = \frac{(\eta - 1) a_0 \Delta a \Delta T}{\kappa I}, \]

\[ \frac{a \xi}{\rho \kappa I \eta^2} \frac{\partial T_I}{\partial \eta} = \frac{\rho a_0^2 \Delta a \Delta T}{\kappa I t_a d_I}, \]

\[ \frac{12 \mu \omega^2}{\kappa_I \eta^6} = \frac{10(\Delta a)^2 \mu_I}{t_s^2}, \]

(35)

Here \( \Delta a \) and \( \Delta T \) are the characteristic variations of the bubble size and liquid temperature, \( d_I \) is the thermal boundary layer thickness, and \( t_a \) is the characteristic time. Consider fast bubble oscillations. In this case we have \( \Delta a \sim \epsilon \omega a_I \), \( \Delta T \sim \epsilon T_, \), \( d_I \sim L_{T_I} \), and \( t_a \sim \omega^{-1} \). At \( \epsilon \sim 0.1 \) the latter term in (35) is small compared with the first term for \( \omega < \omega_I = \rho \omega I T / \mu I \). Normally this limitation is not restrictive (e.g., for water and helium at atmospheric pressures we have \( \omega_I / 2 \pi \sim 10^{11} \text{ Hz} \)). The convective term is related to the liquid motion generated by the moving bubble surface (the third term) and condensation or evaporation (the fourth term). The third term (35) is small at small \( \epsilon \) even at high frequencies, since in this case high temperature gradients of order \( \epsilon T_0 / L_{T_I} \) are realized in a thin boundary layer \( \eta - 1 \sim L_{T_I} / a_I \) and \( (\eta - \eta^2) / \eta - \epsilon T_0 \). The fourth term (35) is small compared to the first term (35) if \( \epsilon \rho a_I L_{T_I} \). For \( \epsilon \sim 0.1 \), \( \rho \sim 10^{-3} \), and bubbles of typical radius \( a_I \sim 1 \text{ mm} \), the thickness of the thermal boundary layer in the liquid should be much larger than \( 0.1 \mu \text{ m} \). For water at atmospheric pressures this limits the theory to frequencies \( \omega / 2 \pi \sim 1 \text{ MHz} \). Note that the acoustic wavelength is of order 1 mm at frequencies \( \omega / 2 \pi \sim 1 \text{ MHz} \), and the theory is limited by such frequencies for 1 mm bubbles.

We assume that all functions in (32)-(34) are periodic with respect to \( t_0 \). For example,

\[ u_m(\eta, t_0, t_1, \ldots) = \text{Re} \left\{ \sum_{n=0}^{m} u_{mn}(\eta, t_1, t_2, \ldots) e^{i n t_0} \right\}, \]

\[ p_m(t_0, t_1, t_2, \ldots) = \text{Re} \left\{ \sum_{n=0}^{m} p_{mn}(t_1, t_2, \ldots) e^{i n t_0} \right\}, \]

where \( u_{mn} \) and \( p_{mn} \) are the complex amplitudes (the number of modes increases with the number of approximations because of nonlinear generation of subharmonics).

We should also notice that the first two orders of approximation of the present theory does not depend on initial conditions. This assumes that the bubble resides in the liquid for long time and a quasisteady slowly changing average temperature profile is developed near the bubble. In the \( m \)th order approximation equation (32) written for zero mode

\[ \frac{1}{\eta^2} \frac{\partial}{\partial \eta} \left( \eta^2 \frac{\partial u_{m0}}{\partial \eta} \right) = -f_{m0} \]

shows that the nonstationary, \( \chi(\partial u_{m0} / \partial t_k) \), and convective, \( Pe_0(\eta - \eta^2)(\partial u_{m0} / \partial \eta) \), terms are considered to be small compared to the conductive term \( (1/\eta^2)(\partial / \partial \eta) \chi \left[ \eta^2 (\partial u_{m0} / \partial \eta) \right] \) and are passed to the approximations of order higher than \( m \). Here \( Pe_0 = (\epsilon^2 a_0 / L_{T_0}^2) (\partial a_0 / \partial t_k) \) is the Peclet number of the slow bubble motion, and \( \chi = \epsilon^2 a_0^2 / L_{T_0}^2 \) (we show later that nothing in the present theory depends on \( t_k \), so \( k \equiv 2 \)). This is correct from the point of view of the current formal asymptotic procedure, however, is questionable from the physical viewpoint, because actual values of \( \chi \) and \( Pe_0 \) may not be small. Bringing these terms into the \( m \)th order approximation creates a mathematical problem, which solution currently is not available. Note that Fyrillas and Szeri obtained a solution for transient slowly evolving fields considering the nonstationary term to be of the same order as the conductive term, but for constant \( a_0 (Pe_0 = 0) \). If significant growth or shrinkage of the bubble occur in some time scale \( t_k \) (so \( \partial a_0 / \partial t_k \sim a_0 \)), then in that scale \( Pe_0 \sim \chi \), and the convective term is of the same order of magnitude as the nonstationary term.

The assumption on periodicity of functions in the fast time scale is not valid for initial value problems at times of order of one period of oscillation. Fyrillas and Szeri evaluated that time as a few periods of oscillation. The depth of penetration of temperature perturbations into the vapor and liquid due to oscillations of the bubble surface temperature is much smaller than the thickness of the thermal boundary layer evolving in slow time scales. That is why the stage of establishment of a periodical solution is not important when we consider slowly changing temperature fields. Since slowly changing fields don't depend only on slow variables the initial conditions for their evolution should also be formulated in slow time scales. In the present theory the initial conditions appear in the third-order approximation, which also requires a spatial matching procedure and is described in a separate section.

For each mode Eqs. (32) and (33) are replaced by equations of the same form, but with the factor \( i n \) in place of the operator \( \partial / \partial t_0 \). Solution of these linear problems are available and can be represented in the following form:

\[ u_{mn} = \frac{1}{\eta} \left[ A_{mn} \exp \left[ - (\eta - 1) \frac{a_0}{L_{T_I}} \sqrt{n} \right] + F_{mn}(\eta) \right], \]

(36)

\[ v_{mn} = k_s p_{mn} + \frac{1}{\eta} \left[ B_{mn} \sinh \left( \eta \frac{a_0}{L_{T_I}} \sqrt{n} \right) + G_{mn}(\eta) \right], \]

(37)

\[ F_{mn}(\eta) = \frac{1}{2 \sqrt{n} a_0} \int_{L_{T_I}}^{a_0} \exp \left[ - (\eta - n) \frac{a_0}{L_{T_I}} \sqrt{n} \right] x f_{mn}(x) dx, \]

(38)
$$G_{mn}(\eta) = \frac{1}{\sqrt{i n}} \frac{L_T}{a_0} \int_0^1 \sinh \left( (x - \eta) \frac{a_0}{L_T} \sqrt{i n} \right) x g_{mn}(x) \, dx,$$

$$A_{mn} = T_{mn} - F_{mn}(1),$$

$$B_{mn} = \left[ \sinh \left( \frac{a_0}{L_T} \sqrt{i n} \right) \right]^{-1} \left[ T_{mn} - G_{mn}(1) - k_y p_{mn} \right].$$

Here we consider that $\text{Re} \{ \sqrt{i} \} > 0$.

The complex amplitudes of the heat fluxes can be determined as

$$q_{mn} = H_n T_{mn} - C_{mn}, \quad r_{mn} = I_n (k_y p_{mn} - T_{mn}) + D_{mn},$$

$$H_n = 1 + \frac{a_0}{L_T} \sqrt{i n}, \quad I_n = \frac{a_0}{L_T} \sqrt{i n} \coth \left( \frac{a_0}{L_T} \sqrt{i n} \right) - 1,$$

$$C_{mn} = 2 \frac{a_0}{L_T} \sqrt{i n} F_{mn}(1),$$

$$D_{mn} = \frac{a_0}{L_T} \sqrt{i n} \left[ \sinh \left( \frac{a_0}{L_T} \sqrt{i n} \right) \right]^{-1} G_{mn}(0).$$

**D. Complex amplitudes**

Substituting asymptotic series (21)–(29) and (20) into (14)–(18) and collecting terms for the same powers of $e$, we obtain the following linear inhomogeneous equations to determine the unknowns in the $n$th-order approximation:

$$\frac{\partial a_m}{\partial t_0} - \omega_m - \rho \xi_m = Y_m^{(1)},$$

$$-2L_\sigma a_m + \frac{a_0^2}{L_p} \left( 1 + 4 \frac{L_\mu}{L_C a_0} \right) \frac{\partial}{\partial t_0} + \frac{4L_\mu^2}{a_0^2} w_m - \left( 1 + \frac{a_0}{L_C} \frac{\partial}{\partial t_0} \right) p_m = Y_m^{(2)},$$

$$3\gamma \frac{\partial a_m}{\partial t_0} + \frac{\partial p_m}{\partial t_0} - 3\gamma \xi_m + 3\gamma \frac{L_T^2}{a_0^2} r_m = Y_m^{(3)},$$

$$-\lambda k_y a_0^2 \left( 1 + k_x \left[ 1 - \rho \left( 1 - \frac{2L_\sigma}{a_0} \right) \right] \right) \xi_m - q_m + \lambda r_m = Y_m^{(4)},$$

$$\xi_m + \frac{L_\beta}{a_0} \left[ k_y (1 - \rho) p_m - T_m \right] = Y_m^{(5)},$$

where the right-hand sides, $Y_m^{(j)}$, $j = 1, \ldots, 5$, depend on the low-order approximations.

To obtain equations for the complex amplitude of the $n$th mode of oscillations we can replace $\partial \partial t_0$ in these equations with $i n$. Consequently, Eqs. (43)–(47) and (40) for the complex amplitudes can be represented in the following matrix form:

$$M_n X_{mn} = Y_{mn},$$

$$M_n = \begin{pmatrix}
    in & -1 & 0 & -\rho & 0 & 0 & 0 \\
    -2L_\sigma/a_0 & M_{22} & M_{23} & 0 & 0 & 0 & 0 \\
    3\gamma in & 0 & in & -3\gamma & 0 & 0 & 3\gamma L_T^2/a_0^2 \\
    0 & 0 & 0 & M_{44} & 0 & -1 & \lambda \\
    0 & 0 & M_{53} & 1 & -L_\beta/a_0 & 0 & 0 \\
    0 & 0 & 0 & 0 & H_n & -1 & 0 \\
    0 & 0 & -k_y I_n & 0 & I_n & 0 & 1
\end{pmatrix},$$
where

\[
M_{22} = \frac{a_0^2}{L_p^2} \left( 1 + 4 \frac{L^2}{L C a_0} \right) \text{in} + \frac{4L^2}{a_0^2}, \\
M_{23} = -1 - i a_0 \frac{L}{L C}, \\
M_{44} = -\frac{L^2}{L_T^2} \left[ \lambda \frac{a_0^2}{L^2} + \frac{1}{k_x} \right] + 1 - \rho \left[ 1 - \frac{2L^2}{a_0^2} \right], \\
M_{53} = \frac{(1-\rho)k_x L_p}{a_0}, \quad (49)
\]

\[
Y^{(6)}_{mn} = C_{mn}, \quad Y^{(7)}_{mn} = D_{mn}.
\]

Let us now specify the structure of the right-hand-side terms \(Y_{mn}\). We can represent them as

\[
Y_{mn} = S_{mn} + N_{mn} + F_{mn},
\]

where \(S_{mn}\) is generated by slow time scale evolution of linear terms, \(N_{mn}\) is generated by nonlinear terms, and \(F_{mn}\) is the external forcing. \(S_{mn}\) can be found from linear equations (43)-(47), (32), (33), (38), (39), (41), and (42) and asymptotic series (20) and (21)-(29):

\[
S_{mn}^{(1)} = -\frac{1}{a_0} \left[ a_0 \frac{\partial \phi_0}{\partial t_m} + \sum_{j=1}^{m-1} \frac{\partial (a_0 \phi_{jn})}{\partial t_{m-j}} \right], \quad (51)
\]

\[
S_{mn}^{(2)} = -\frac{a_0^2}{L_p^2} \left[ \frac{L^2}{L C a_0} \sum_{j=1}^{m-1} \frac{\partial \phi_{jn}}{\partial t_{m-j}} - \frac{a_0}{L C} \sum_{j=1}^{m-1} \frac{\partial p_{jn}}{\partial t_{m-j}} \right], \quad (52)
\]

\[
S_{mn}^{(3)} = -\frac{L^2}{L C a_0} \left[ \frac{L^2}{L C a_0} \sum_{j=1}^{m-1} \frac{\partial \phi_{jn}}{\partial t_{m-j}} - \frac{a_0}{L C} \sum_{j=1}^{m-1} \frac{\partial p_{jn}}{\partial t_{m-j}} \right], \quad (53)
\]

\[
S_{mn}^{(4)} = S_{mn}^{(5)} = 0,
\]

\[
S_{mn}^{(6)} = -\frac{a_0^2}{L_T^2} \int_0^{\eta} \sum_{j=1}^{m-1} \frac{\partial \phi_{jn}}{\partial t_{m-j}} \exp \left[ -\left( \eta - 1 \right) \frac{a_0}{L_T} \right] \eta d \eta,
\]

where the superscript in the brackets near \(S_{mn}\) shows the number of the vector \(S_{mn}\) component and \(\delta_{ij}\) is the Kronecker delta.

In our case, \(F_{mn}\) is nonzero only for \(n=0,1\). The order \(m\) for which \(F_{mn} \neq 0\) depends on the relation between the small parameters \(\epsilon \) and \(\delta\) determined by (11) and (13). For \(\delta \sim \epsilon^m\), we have \(F_{mn} \neq 0\) only at \(m=m_0\). Since \(\epsilon \) and \(\delta\) are independent parameters, \(m_0\) can be selected arbitrary. However, a better way is to set \(m_0\) using the characteristic asymptotic form. This can be done by the following physical reasoning. \(\epsilon\) and \(\delta\) are parameters responsible for two instabilities regarding a vapor bubble. The former is responsible for the instability due to rectified heat transfer and the latter for the instability due to the surface tension and deviation from the saturation state. The characteristic times required for development of these two instabilities are proportional, respectively, to the energy of the acoustic field, or \(\epsilon^2\), and to the deviation from the equilibrium state, or \(\delta\). These two instabilities can be brought in the same order of approximation if

\[
\delta \sim \epsilon^2.
\]

This is a condition for the characteristic asymptotic form. Accepting (52) we have the following expressions for the nonzero components of \(F_{mn}\):

\[
F_{20}^{(2)} = -\frac{1}{\epsilon^2} \left[ \Delta + \frac{2L^2}{a_0} \right], \quad F_{11}^{(2)} = -\left[ 1 + i a_0 \right] \frac{L}{L C}.
\]

An expression for \(N_{mn}\) can also be derived using governing equations and asymptotic representations. We do not reproduce them in general form here since they are too unwieldy.

### E. Evolution in slow time scales

To obtain equations describing evolution of the unknowns in slow time scales, let us consider the operator \(M_0\),

\[
M_0 = \begin{pmatrix}
0 & -1 & 0 & -\rho & 0 & 0 & 0 \\
-2L_\sigma /a_0 & 4L^2 /L_p^2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -3\gamma & 0 & 0 & 3\gamma L^2 /L_T^2 \eta^2 /a_0^2 \\
0 & 0 & 0 & 0 & -\rho & 0 & 0 \\
0 & 0 & 0 & M_{44} & 0 & -1 & \lambda \\
0 & 0 & 0 & M_{53} & -L_\beta /a_0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
S_{mn}^{(7)} = -\frac{a_0^2}{L_T^2} \int_0^{1} \left[ \sum_{j=1}^{m-1} \frac{\partial \phi_{jn}}{\partial t_{m-j}} - k_\gamma \sum_{j=1}^{m-1} \frac{\partial p_{jn}}{\partial t_{m-j}} \right]
\times \frac{\sinh[(a_0/L_T) \sqrt{\eta}]}{\sinh[(a_0/L_T) \sqrt{\eta}]} \eta d \eta,
\]
The determinant of this matrix is nonzero:

$$\det \mathbf{M}_0 = \frac{6 \gamma (1 - \rho) kL_{\beta}L_\sigma}{a_0^2}.$$  (55)

From (52) we have $L_\sigma/a_0 \sim \delta - \epsilon^2 \approx 1$, or det $\mathbf{M}_0 = O(\epsilon^2)$. According Cramer’s rule the zero-mode components of solution, $X_{m0}$, are

$$X_m^{(i)} = \frac{\det \mathbf{M}_{0m}^{(i)}}{\det \mathbf{M}_0}, \quad i = 1, \ldots, 7,$$  (56)

where $\mathbf{M}_{0m}^{(i)}$ is the matrix $\mathbf{M}_0$ with the $i$th column replaced by the right-hand side vector, $\mathbf{Y}_m$. By definition of an asymptotic expansion $X_m^{(i)}$ should be of order of unity. Thus

$$\det \mathbf{M}_{0m}^{(i)} = O(\epsilon^2), \quad i = 1, \ldots, 7.$$  (57)

The form of matrix $\mathbf{M}_0$ (54) shows that this condition automatically holds for $i = 2, \ldots, 7$. However, for $i = 1$ it becomes nontrivial. Calculating det $\mathbf{M}_0^{(1)}$, we have

$$\det \mathbf{M}_{0m}^{(1)} = -3 \gamma M_{53} \left( \frac{4 \rho L_\mu^2}{L_p^2} Y_{m0}^{(1)} + \frac{3 \gamma L_\beta}{a_0} (Y_{m0}^{(4)} - \frac{4 \rho L_\mu^2}{L_p^2} M_{53} - \frac{L_\beta}{a_0} M_{44}) \right) \times \left( Y_{m0}^{(3)} - \frac{3 \gamma L_\mu^2}{a_0^2} Y_{m0}^{(7)} - \frac{3 \gamma}{a_0} Y_{m0}^{(5)} \right),$$

where $Y_{m0}^{(6)}$ and $Y_{m0}^{(7)}$ can be found as limits of (38), (39), and (41) at $n \to 0$:

$$Y_{m0}^{(6)} = c_{m0} \int_1^\infty \eta f_{m0}(\eta) d\eta,$$

$$Y_{m0}^{(7)} = d_{m0} \int_0^1 \eta^2 g_{m0}(\eta) d\eta.$$  (58)

To avoid secular terms in expansion (21), $a_{m0}$ and det $\mathbf{M}_{0m}^{(1)}$ should be limited at $t_1 \to \infty, j = 1, 2, \ldots$. This is the only requirement for the uniform asymptotic expansion which is not unique. The nonuniqueness of the asymptotic expansion can be easily demonstrated for the case of $L_\sigma = 0$ (the effect of surface tension is neglected). In this case det $\mathbf{M}_0 = 0$, det $\mathbf{M}_{0m}^{(1)} = 0$, and $a_{m0}$ is determined as an arbitrary limited function of slow times. If we set $a_{m0} = 0$, for $m = 1, 2, \ldots$, then we uniquely select the expansion in which $a_0$ is the bubble radius averaged over the period of oscillation, $\langle a \rangle$, since $\langle a \rangle = a_0 (1 + \epsilon a_0 + \epsilon^2 a_2 + \cdots)$. Such a definition of arbitrary $a_{m0}$ was used in a previous investigation. 24 Similarly, for the case $L_\sigma \neq 0$ we can specify

$$\det \mathbf{M}_{0m}^{(1)} = 0.$$  (59)

According to (56) this leads to

$$a_{m0} = 0, \quad \text{for} \ m = 1, 2, \ldots, \quad \langle a \rangle = a_0,$$  (60)

which is a convenient definition of $a_0$.

**IV. FIRST-ORDER APPROXIMATION**

In the first-order approximation the nonlinear term $\mathbf{N}_{1n}$ in (50) is zero, the components of the external forcing are given by (53), while the slow-scale evolution vector $\mathbf{S}_{1n}$ has only two nonzero components (51):

$$S_{10}^{(1)} = -\frac{1}{a_0} \frac{\partial a_0}{\partial t_1}, \quad S_{10}^{(3)} = -\frac{3 \gamma}{a_0} \frac{\partial a_0}{\partial t_1},$$  (61)

Using (53) we can explicitly resolve (48) for the first mode using Cramer’s rule:

$$X_{11}^{(j)} = \left( 1 + \frac{ia_0}{L_C \det \mathbf{M}_1} \right)^{-1} \Delta^{(j)}(j = 1, 2, \ldots, 7),$$  (62)

where $X_{11}^{(j)}$ are the components of the vector $\mathbf{X}_{11}$ and $\Delta^{(j)}$ can be represented as

$$\Delta^{(1)} = \frac{L_\beta}{a_0} (i M_{44} + 3 \gamma k \lambda I_1) - (i + 3 \gamma M_{53}) (H_1 + \lambda I_1),$$

$$\Delta^{(2)} = i \Delta^{(1)} - \rho \Delta^{(4)}, \quad \Delta^{(3)} = 3 i \gamma \left( H_1 + \lambda I_1 - \frac{L_\beta}{a_0} M_{44} \right),$$

$$\Delta^{(4)} = 3 i \gamma \left( \frac{k_L L_\beta}{a_0} I_1 - M_{53} (H_1 + \lambda I_1) \right),$$

$$\Delta^{(5)} = 3 i \gamma \left( k_L L_\beta \lambda - M_{53} M_{44} \right),$$

$$\Delta^{(6)} = H_1 \Delta^{(5)}, \quad \Delta^{(7)} = I_1 (k_L \Delta^{(3)} - \Delta^{(5)}),$$

$$\det \mathbf{M}_1 = \frac{a_0^2}{L_p^2} \left[ 1 + \frac{4 L_\mu^2}{L_C a_0} + \frac{4 L_\mu^2}{a_0^2} \right] \Delta^{(2)} - \left( 1 + \frac{4 i a_0}{L_C} \right) \Delta^{(3)}.$$  (63)

Note that these equations are consistent with previous results. 11,21,17,38,24

From (57) and (59), we have, using (50), (53), (61), and (49),

$$\frac{\partial a_0}{\partial t_1} = 0, \quad a_0 = a_0(t_2, t_3, \ldots).$$  (64)

To determine a slow-time evolution of the mean bubble radius we need to consider the second-order approximation.

**V. EQUATION FOR RECTIFIED HEAT TRANSFER IN THE SECOND-ORDER APPROXIMATION**

Since $a_0$ does not depend on $t_1$, the same is true for all other variables. Thus, the nonzero components of vector $\mathbf{S}$ in the second-order approximation are

$$S_{20}^{(1)} = -\frac{1}{a_0} \frac{\partial a_0}{\partial t_2}, \quad S_{20}^{(3)} = -\frac{3 \gamma}{a_0} \frac{\partial a_0}{\partial t_2},$$  (65)

The nonzero forcing term is (53), while the components of the nonlinear term $\mathbf{N}_{2n}$ can be found in the second-order approximation.
approximation from quadratic nonlinearities. The nonzero components include modes 0 and 2 which correspond to the average fluxes and nonlinear doubling of the oscillation frequency. To obtain the equation of rectified heat transfer it is sufficient to consider just zero-mode vector \( \mathbf{N}_{20} \). After some algebra one can find the components of this vector in the following form:

\[
N_{20}^{(1)} = 0,
\]

\[
N_{20}^{(2)} = \text{Re} \left\{ \frac{4L^2 + ia^2}{2L_p} \ln \frac{a_0^2 + 2a_0}{4L^2} \right\} w_{11}^2
\]

\[
- \frac{L}{a_0} |a_{11}|^2 - \rho(1 - \rho)a_0^2|\xi_{11}|^2,
\]

\[
N_{20}^{(3)} = \frac{3}{2}\pi
\]

\[
\left\{ \left( 1 - \frac{2}{3}\gamma \right) ip_{11}a_{11}^* + \xi_{11}a_{11}^* + \xi_{11}^*T_{11} \right\},
\]

\[
N_{20}^{(4)} = \frac{\lambda k_a}{2L_T} \text{Re} \left\{ T_{11}^* \xi_{11} - \rho p_{11} \xi_{11}^* \right\}
\]

\[
+ \left( \frac{k}{k_a} + 1 - \rho \right) a_{11}^* e_x^* \right),
\]

\[
N_{20}^{(5)} = \frac{L}{a_0} k_a (1 - \rho) \eta_n^* T_{11}^2 - \frac{1}{4} \text{Re} \left\{ \xi_{11}^* T_{11} \right\},
\]

\[
N_{20}^{(6)} = C_{20} = \frac{a_0^2}{2L_T^3} \text{Re} \left\{ \left[ i p_{11} a_{11}^* - \xi_{11} a_{11}^* \right.ight.
\]

\[
\left. \left. + \xi_{11}^* T_{11} \right\} \right
\]

Here an asterisk denotes complex conjugate, \( E(z) \) is the third-order integral exponent

\[
E(z) = \int_1^\infty \frac{1}{\eta^2} e^{-z(\eta-1)} d\eta,
\]

and \( \eta_n^* \) is the dimensionless second derivative of the saturation pressure along the saturation curve. Using the Clapeyron–Clausius equation (6) for perfect gas, we have

\[
\eta_n^* = \frac{T^2 d^2 p}{\rho_d d T} \left. \right|_{T = T_x}
\]

\[
= 1 - k_s (1 - \rho)(2 - \rho)
\]

\[
\frac{k_s (1 - \rho)^2}{k_x (1 - \rho)}.
\]

From (50), (53), (65), (57), and (66) we can obtain the following equation for rectified heat transfer:

\[
\frac{\partial a_0}{\partial t} = W_2(a_0), \quad V(a_0) = \left[ 1 + \frac{4(1 - \rho)L^2}{L_p^2} M_{53}(a_0) \right.
\]

\[
- \frac{L}{a_0} M_{44}(a_0) \right]^{-1},
\]

\[
W_2(a_0) = a_0 V(a_0) \left[ \frac{1}{a_0} M_{53} \right.
\]

\[
(\Delta + \frac{2L}{a_0}) + M_{53} N_{20}^{(2)}
\]

\[
- \frac{L}{a_0} M_{44} \left[ \frac{1}{a_0} N_{20}^{(3)} \right.
\]

\[
\left. + \frac{2L^2}{a_0^2} N_{20}^{(7)} \right] + N_{20}^{(5)}.
\]

This equation agrees in limiting cases with the equation for rectified diffusion obtained using simplified models. 21,17,24

Note, that Eq. (4.2) in Ref. 24 contains misprints.

VI. THIRD ORDER APPROXIMATION

It is noteworthy that the used asymptotic scheme does not provide uniformly valid expansions in orders higher than 2. The source of asymptotic singularity is connected with the infinite spatial region for the thermal problem in the liquid. Below we consider this problem in detail.

A. Spatial matching of asymptotic expansions

The above procedure for solution of the thermal problem is correct for determination of the \( n \)th modes of fluxes when \( n \neq 0 \). At \( n = 0 \) the temporal derivatives with respect to \( t_0 \) in the left-hand side parts of Eqs. (32) and (33) are zero. In this case the temporal derivative is treated in a straightforward way which causes a singularity of the asymptotic expansion (small parameter near the highest derivative). In the slow time scales a temperature perturbation propagates from the bubble to the bulk of liquid to characteristic distances \( \eta \gg 1 \) and the slow spatial scales should be considered. For the boundary value problem matching of inner and outer asymptotic expansions is an appropriate asymptotic procedure.

The inner problem for \( u_{m0} \) follows from (32):

\[
\eta_1 \frac{\partial}{\partial \eta} \left( \eta^2 \frac{\partial u_{m0}}{\partial \eta} \right) = -f_{m0}, \quad u_{m0}|_{\eta = 1} = T_{m0},
\]

\[
u_{m0}|_{\eta = \infty} = \vartheta_{m0},
\]

where \( \vartheta_{m0} \) is the matching constant to be determined. Solution of this problem is

\[
u_{m0} = \vartheta_{m0} + \int_1^\infty xf_{m0}(x) dx + \int_1^\infty \left. \frac{1}{\eta} T_{m0} - \vartheta_{m0} \right|_1^\infty
\]

\[
- \int_1^\infty xf_{m0}(x) dx + \left. \int_1^\infty x^2 f_{m0}(x) dx \right|_1^\infty
\]

and for zero mode of the heat flux we have
\[ q_{m0} = -\frac{\partial u_{m0}}{\partial \eta} \bigg|_{\eta=1} = T_{m0} - \vartheta_{m0} - \int_{1}^{\infty} \eta f_{m0}(\eta) d\eta. \]  

(70)

Comparing with (40) we can determine
\[ C_{m0} = \vartheta_{m0} + \int_{1}^{\infty} \eta f_{m0}(\eta) d\eta. \]  

(71)

For simplicity we limit ourselves to the third-order approximation requiring a one-term outer expansion. In this case it is enough to introduce one slow variable \( \eta_2 = \epsilon \eta \) (otherwise we introduce for each scale, \( t_k \), its own slow spatial variable, \( \eta_k = e^{\epsilon t_k} \eta, \ k = 2, 3, \ldots \) and consider a zero-order approximation, \( u_0^{(o)} \), to outer solution \( u^{(o)} \) (otherwise we consider a series, \( u^{(o)} = u_0^{(o)} + e u_1^{(o)} + \cdots \)). The outer problem in slow scale \( (t_2, \eta_2) \) is therefore
\[ \frac{a_0}{L_{T_2}^2} \frac{a_0}{\partial t_2} - \eta_2 \frac{\partial a_0}{\partial t_2} \frac{\partial u_0^{(o)}}{\partial \eta_2} - \frac{1}{\eta_2^2} \frac{\partial}{\partial \eta_2} \left( \eta_2^2 \frac{\partial u_0^{(o)}}{\partial \eta_2} \right) = 0. \]  

(72)

Solution of this problem with the initial and boundary conditions
\[ u_0^{(o)}|_{t_2=0} = 0, \quad u_0^{(o)}|_{\eta_2=\infty} = 0, \]  

(73)

can be represented in the form
\[ u_0^{(o)} = \frac{e^3 L_{T_2}}{a_0(t_2) \eta_2} \Phi_{20}|_{t_2=0} \text{erfc} \left( \frac{a_0(t_2, \ldots) \eta_2}{2 L_{T_2} \sqrt{t_2}} \right) \]
\[ + \int_{0}^{t_2} \frac{\partial \Phi_{20}}{\partial \zeta} \text{erfc} \left( \frac{a_0(t_2, \ldots) \eta_2}{2 L_{T_2} \sqrt{t_2 - \zeta}} \right) d\zeta, \]  

(74)

where \( \Phi_{20} \) is an arbitrary function of time.

To match this solution with the inner solution, we substitute \( \eta_2 = \epsilon \eta \) and expand this expression at \( \epsilon \to 0 \):
\[ u_0^{(o)} = \frac{e^3 L_{T_2}}{a_0 \eta} \Phi_{20} - \frac{e^3}{\sqrt{\pi}} \frac{\Phi_{20}|_{t_2=0}}{\sqrt{t_2}} + \int_{0}^{t_2} \frac{\partial \Phi_{20}}{\partial \zeta} \frac{d\zeta}{\sqrt{t_2 - \zeta}} + \cdots \]  

(75)

In the same time the two-term outer expansion of two terms of inner solution, \( u_0^{(i)} \), (69) is
\[ u_0^{(i)} = e^2 u_{20} + e^3 u_{30} + \cdots \]
\[ = e^2 \left\{ \vartheta_{20} + \frac{1}{\eta} \left[ T_{20} - \vartheta_{20} + \int_{1}^{\infty} x(x-1) f_{20}(x) dx \right] \right\} \]
\[ + e^3 \vartheta_{30} + \cdots \]  

(76)

Here we used the fact that \( f_{30} = 0 \). This is true for harmonic oscillations, since the regular nonlinearity of equations generates modes 0 and 2 in the second-order and modes 1 and 3 in the third-order approximation.

Comparing (75) and (76) one can find
\[ \vartheta_{20} = 0, \quad \Phi_{20} = \frac{a_0}{L_{T_2}} (T_{20} + E_{20}), \]  

(77)

\[ \vartheta_{30} = -\frac{1}{\sqrt{\pi}} \left[ \frac{\Phi_{200}}{\sqrt{t_2}} + \int_{0}^{t_2} \frac{\partial \Phi_{200}}{\partial \zeta} \frac{d\zeta}{\sqrt{t_2 - \zeta}} \right], \]
\[ \Phi_{200} = \Phi_{20}|_{t_2=0}. \]  

(78)

**B. Equation for rectified heat transfer**

In the third-order approximation we have
\[ Y_{30}^{(1)} = -\frac{1}{a_0} \frac{\partial a_0}{\partial t_3}, \quad Y_{30}^{(2)} = 0, \quad Y_{30}^{(3)} = -3 \frac{\partial a_0}{a_0} \frac{\partial t_3}{\partial t_3}, \]
\[ Y_{30}^{(4)} = Y_{30}^{(5)} = C_{30} = \vartheta_{30}, \quad Y_{30}^{(7)} = 0. \]  

General condition (59) and (57) provides us with the following equation:
\[ \frac{\partial a_0}{\partial t_3} = W_{3}, \quad W_{3} = L_{T} V(a_0) \vartheta_{30}. \]  

(79)

Zero mode of temperature in the second-order approximation can be found using (54)-(56), (50), (65), and (67):
\[ T_{20} = C_{20} + \lambda D_{20} - N_{20}^{(4)} \]
\[ + M_{44} \left( \frac{1}{a_0} W_{2}(a_0) + \frac{L_{T}^2}{a_0^2} D_{20} - \frac{1}{3} \gamma N_{20}^{(3)} \right). \]  

(80)

We also can determine
\[ E_{20} = \frac{1}{2} \text{Re} \left\{ a_{11}^{*} T_{11} \left[ 1 + a_{11}^{*} \eta^2 \right] + \frac{a_{0}^{2}}{L_{T_1}} E_{20} \left[ a_{0}^{*} \eta^2 \right] \right\} \]
\[ + \frac{a_{0}^{2}}{2L_{T}} |w_{11}|^2. \]  

(81)

Equations (77), (80), and (81) specify \( \Phi_{20} \) as a function of \( a_0 \).

Generally, if we know equations in time scales \( t_2, t_3, \ldots \) then we can introduce one slow time, \( \tau = e^{2 \epsilon t_0} \), and represent equation for rectified heat transfer in the form
\[ \frac{\partial a_0}{\partial \tau} = \frac{\partial a_0(t_2, t_3, \ldots)}{\partial t_2} + e \frac{\partial a_0(t_2, t_3, \ldots)}{\partial t_3} + \cdots \]
\[ = W_{2}(a_0, \tau, \epsilon \tau, \ldots) + \cdots \]  

(82)

Thus, we have for \( a_0 \) the following equation:
\[ \frac{\partial a_0}{\partial \tau} = W_{2}(a_0) - \frac{e L_{T} V(a_0)}{\sqrt{\pi}} \Phi_{200} \left( \frac{\partial a_0}{\partial \zeta} \right) \frac{d\zeta}{\sqrt{\tau - \zeta}} + O(\epsilon^2). \]  

(83)

The term \( W_{3} \) reflects an important physical effect which shows that the growth rate depends not only on the current value of the bubble radius, but also on the history of the development of the thermal boundary layer. Mathematically
the type of equation also changes from an ordinary differential equation in the second-order approximation to a nonlinear integrodifferential equation in the third-order approximation.

C. The steady radius and its stability

There exists a possibility of stabilization of vapor bubble radius in an acoustic field. Complete stability analysis cannot be performed based on asymptotic theory, since the residual terms, such as in Eq. (83) can cause a growth or decay of bubble radius in slower time scales than that taken into account (secular terms). However, if the steady radius exists, then it is stable in any slow time scale. If it does not exist then there is no stable radius in the low-order approximations. That is why low-order approximations can provide valuable information on the bubble stability. Everywhere below when we speak about stability we mean stability within the framework of the third-order theory, which can potentially be violated in slower time scales.

Let us define the steady radius, \( a_\ast \), as a zero of function \( W_2 \):

\[
W_2(a_\ast) = 0.
\]

Consider a small perturbation of \( a_0 \) near this radius:

\[
\alpha = a_0 - a_\ast.
\]

Linearizing (83) near \( a_0 = a_\ast \) we find

\[
\frac{\partial \alpha}{\partial \tau} = W_\ast \alpha - \frac{e L \beta V_\ast}{\sqrt{\tau}} \left( \Phi_{200} + \Phi_{200}^\prime \int_0^\tau \frac{\partial \alpha(\xi)}{\partial \xi} d\xi \right),
\]

\[
W_\ast = \frac{dW_2(a_0)}{da_0} \bigg|_{a_0 = a_\ast}, \quad \Phi_{200}^\prime = \frac{d\Phi_{200}(a_0)}{da_0} \bigg|_{a_0 = a_\ast},
\]

\[
V_\ast = V(a_\ast).
\]

Solution of this equation can be obtained using the forward and inverse Laplace transforms, and can be represented in the form:

\[
\alpha(\tau) = b_1 e^{z_1^\tau} \text{erfc}(z_1 \sqrt{\tau}) + b_2 e^{z_2^\tau} \text{erfc}(z_2 \sqrt{\tau}),
\]

where \( b_1 \) and \( b_2 \) are constants depending on initial conditions, while \( z_{1,2} \) are the roots of the characteristic equation:

\[
z_{1,2} = -\frac{1}{2} e L \beta V_\ast \Phi_{200}^\prime \pm \sqrt{\frac{1}{2} e L \beta V_\ast \Phi_{200}^\prime + W_\ast}.
\]

If \( W_\ast > 0 \), then both roots \( z_1 \) and \( z_2 \) are real and have opposite signs. Since function \( e^{z \text{erfc}(z)} \) exponentially grows at real positive \( z \to \infty \) the steady radius \( a_\ast \) is unstable. This conclusion is consistent with the result of the second-order theory [if we set \( \epsilon = 0 \) in (86)]. If \( W_\ast < 0 \) the second-order theory shows that \( a_\ast \) is stable. In the third-order theory the stability criterion is \( W_\ast < -\frac{1}{2} e L \beta V_\ast \Phi_{200}^\prime \). This provides \( |\arg z_{1,2}| > \pi/4 \), necessary for exponential decay of \( \alpha(\tau) \). In the case \( -\frac{1}{2} e L \beta V_\ast \Phi_{200}^\prime < W_\ast < 0 \), one can check using the criterion \( |\arg z_{1,2}| > \pi/4 \) that \( a_\ast \) is stable at \( \Phi_{200}^\prime > 0 \) and \( a_\ast \) is unstable at \( \Phi_{200}^\prime < 0 \).

The third-order approximation provides the following asymptotic expansion near the stable radius at large times:

\[
a_0(\tau) = a_\ast + \frac{e L \beta V_\ast}{W_\ast} \Phi_{200}^\prime \int_0^\tau \frac{\partial \alpha(\xi)}{\partial \xi} d\xi + \left( \frac{V_\ast}{W_\ast} - \frac{1}{2} W_\ast \right) \frac{\sqrt{\pi}}{\tau} + O(\tau^{-3/2}).
\]

VII. NUMERICAL RESULTS AND DISCUSSION

Computations for water and helium of vapor bubble dynamics in acoustic fields were carried out for a range of bubble sizes, frequencies, and amplitudes appropriate for the present theory. Since water and helium vapors deviate from the perfect gas behavior (particularly true for helium at low temperatures) the property values given in Table I were utilized in computations. Other quantities, such as the gas constant and the specific heat ratio, were derived using the perfect gas relations.

Figures 1 and 2 demonstrate typical dependences of the amplitude, \( |a_1| \), and phase, \( \arg (a_1) \), of bubble radius oscillations on the average bubble radius, \( a_0 \). Computations were

![FIG. 1. Relative amplitude of the forced vapor bubble radial oscillation in a 60 kHz acoustic field. The numbers near the curves show the values of the accommodation coefficient, \( \beta \). The curve marked as “equilibrium” is computed using the quasiequilibrium scheme of phase transition.](image-url)
performed using (62) and (63) for water at 1 atm and an acoustic frequency 60 kHz. Qualitatively the dependence is the same for water at different pressures and frequencies or for other liquids. There exist two limiting cases: the case of equilibrium (or more precisely quasiequilibrium) phase transitions where we assume that the vapor pressure is equal to the saturation pressure at the interface temperature, and the case of the absence of phase transitions where \( \beta = 0 \). The equilibrium case can be formally obtained if we set \( \beta = \infty \) despite the actual value of \( \beta \) cannot exceed 1. The computed curves show that the linear response of the vapor bubble to acoustic excitation strongly depends on \( \beta \). The two limiting cases determine two resonances. For \( \beta = 0 \) we have only the primary resonance due to the liquid inertia and the vapor elasticity typical for gas bubbles. In the case of equilibrium phase transitions an additional low-frequency ("second") resonance due to phase transition and surface tension takes place. For water vapor bubbles at moderate \( \beta \) the amplitude of oscillation at the primary resonance is smaller than at the second resonance. For small \( \beta \) the bubble response curves show strong primary resonance. Note that at \( \beta = 0 \) formally there exist the "second" resonance at \( a \approx 2 \sigma / (3 \rho_s) \). However, to reach this resonance we need to have \( \Delta p = -3 \rho_s \), which corresponds to negative liquid pressure. This fact was discussed earlier by Khabeev.\(^{11}\) Moreover, results for small bubble sizes below the theory limit line in Figs. 1 and 2 violate assumption (13) and are not justified by the present theory.

Figure 3 illustrates the temperature profiles inside and outside the bubble at a fixed moment of time \( (t_0 = 2 \pi n) \). Computations were made for 50 \( \mu \)m bubbles using analytical solutions (36) and (37). All parameters were the same for all plotted curves, except for the value of the accommodation coefficient. It is seen that in the illustrated cases, \( \beta \) substantially influences the temperature distribution in the vapor. The influence of \( \beta \) on the temperature in the liquid is less, and there is no visible difference between computations with \( \beta = 0.04 \) and \( \beta = \infty \). At smaller \( \beta \), the temperature gradients in the vapor are much higher than those predicted by the quasiequilibrium theory, and the heat flux in the vapor can be comparable with the heat flux in the liquid. This affects the interface temperature and the temperature profiles in the liquid.

Even for a fixed substance and ambient conditions, such as for water at 1 atm ambient pressure and 100 °C, classification of the vapor bubble behavior in acoustic fields is difficult, since three parameters of the pressure field \( \omega, P_A, \) and \( \Delta p \), the initial radius of the bubble introduced into the field, \( a_{th} \), and the unknown \( \beta \) form a five-dimensional parameter space. First we consider \( \Delta p = 0 \). It is known from previous studies and computations\(^{3,62}\) that in this case there exist two steady average radii of the vapor bubble, \( a_{th} \), and \( a_g \).

The lower steady radius is unstable and is known as the threshold radius, \( a_{th} \), since bubbles with initial size \( a_{in} < a_{th} \) collapse, while bubbles with \( a_{in} > a_{th} \) grow in the acoustic field. The upper steady radius, \( a_{st} \), is stable if the stability criterion obtained in the preceding section holds and for bubbles with \( a_{in} > a_{th} \) their radii tend to \( a_{st} \) at large times. This is seen from the phase portrait of Eq. (67) plotted in Fig. 4. The growth rate, \( da_0/dt = e^2 \partial a_0 / \partial \tau \), is zero at \( a_0 = a_g \). In the illustrated case the maximum growth rate is realized at small and high (of order 1) values of \( \beta \).

In the case of small \( \beta \) the growth rate has a sharp peak due to the phase shift and a high amplitude of the resonance bubble oscillations (see Figs. 1 and 2) producing nonlinear effects. However, for \( \beta = 0 \) the rate of evaporation/condensation is zero and the growth rate is zero. That is why the growth rate at very small values of the accommodation coefficient, such as \( \beta = 0.0001 \) in Fig. 4 is smaller than for \( \beta = 0.001 \). Such a nonmonotonic dependence of the growth rate on the value of the accommodation coefficient was noticed earlier.\(^{24}\) The case of high \( \beta \) demonstrates that the threshold and stable radius can be substantially smaller than...
the primary resonance radius, since they are determined by the second resonance (see Fig. 1). For an acoustic frequency of 60 kHz and $\beta$ from 0.01 to 0.1 the maximum of the growth rate is located between the two resonance bubble sizes.

Figure 5 shows the relation between $a_*$ and the sound amplitude at the bubble location for various $\beta$. For each $\beta$ there exists a minimum of the plotted dependence which determines the amplitude threshold of vapor cavitation $P_A^{(\min)}$. For low intensity sound with $P_A < P_A^{(\min)}$ any bubble will collapse due to the effect of surface tension. For $P_A > P_A^{(\min)}$ two steady radii: unstable, $a_{th}$ (plotted by thin lines) and stable, $a_{st}$ (plotted by thick lines), can be found from Fig. 5. Here and everywhere below stability of the steady radius was determined using the stability criterion in the third-order approximation. However we should notice that in the computed cases this criterion gives practically the same results as the stability criterion in the second-order approximation and the transition from stable to unstable radius occurs in very close vicinity of the extrema of the curves.

Figure 6 shows the same dependence, but for three different frequencies. Note that although the present theory can be applied to high frequency ultrasonic fields it is limited by relatively small amplitudes (weakly nonlinear approximation, $\epsilon \ll 1$). For water at atmospheric pressure and $\Delta p = 0$ the theory is limited to frequencies of order 100 kHz and less. For other substances such as cryogenic liquids the weakly nonlinear approximation is valid for much higher frequencies. Figure 7 shows that for liquid helium near the boiling point at atmospheric pressure, frequencies of 10 MHz and higher can be described by the present theory. Computations for liquid helium using the equilibrium scheme of phase transitions and the nonequilibrium scheme with $\beta = 1$ recommended at temperatures above the lambda-point did not show a noticeable difference for frequencies up to 10 MHz.

Note that for large bubbles at high frequencies additional roots, $a_{ac}$, of function $W_2(a_0)$ appear. One such root is plotted by the light gray line in Fig. 7. An analysis of the computed cases show that for these roots $\omega a_{ac} \sim C$, which violates the assumption $\omega a \ll C$ required by the present theory, and they are not plotted in Figs. 6 and 7 for other curves.

The curve for the steady radius at 1 kHz plotted in Fig. 6 shows that the stable radius of vapor bubble for $P_A \approx 0.1$ atm is of order 10 cm and larger for higher $P_A$. This explains why in the numerical simulations of Hao and Prosperetti...
performed for frequencies of about 1 kHz, the stable radius was not achieved. The computations were carried out for millimeter bubbles which are much smaller than the stable size and should grow even at very slow growth rates. A site is clear from Fig. 6 for higher frequencies stable oscillations can be reached for millimeter bubble sizes for 10 kHz fields and submillimeter bubbles for higher frequencies.

Positive values of $D_p$ correspond to subcooled liquids. As seen in Fig. 7 for higher $D_p$ higher intensity acoustic fields are required for acoustic vapor cavitation. For high $\beta$ additional roots of the function $W_2(a_0)$ can appear, which corresponds to two unstable and two stable radii related to the primary and to the second resonances. This is illustrated in Fig. 8 where the plotted horizontal line intersects the curve for $\beta=1$ four times. In the illustrated case for $\beta$ smaller than 0.1 only one stable radius corresponding to the primary resonance is realized.

Negative values of $D_p$ correspond to superheated liquids. In such liquids bubbles with radii $a_{in}>a_e$, where $a_e$ is determined by (12) grow in the absence of an acoustic field. As shown in Fig. 9 acoustic fields of relatively low frequency shift the threshold of bubble growth toward lower sizes. In the illustrated case of 10 kHz for water at atmospheric pressure and $\Delta p=-0.01 \text{ atm} (a_e=116 \mu m)$ there is no steady radius, and bubbles grow indefinitely if they exceed the vapor cavitation threshold. It is interesting that for higher frequencies at the same conditions there can appear three steady radii (one stable radius and two thresholds) in the range $a<a_e$. In the case of 60 kHz driving frequency illustrated in Fig. 10, curves computed at various values of $\beta$ show local minima and maxima at $a<a_e$.

Figure 10 shows that there exists a qualitative difference between the bubble dynamics at low, moderate, and high $\beta$. For example, for an acoustic amplitude $P_A=0.15 \text{ atm}$ indicated by the dotted line at $\beta=0.04$ bubbles with radii $a_{st} > 29 \mu m$ will unlimittedly grow, while for $\beta=0.1$ bubbles with initial radii $27 \mu m<a_{th}<82 \mu m$ will stabilize near $a_{st}=51 \mu m$ and only bubbles with $a_{in}>82 \mu m$ will grow unlimittedly. The same situation takes place for $\beta=0.01$, but with threshold radii, $a_{th}^{(1)}=32 \mu m$ and $a_{th}^{(2)}=98 \mu m$, and the stable radius $a_{st}=89 \mu m$. Such influence of the accommodation coefficient on the threshold and stable radii of vapor bubbles in acoustic fields can be exploited for measurement of $\beta$.

This is supported by Figs. 11–12 which demonstrate the bubble dynamics at different values of the accommodation coefficient with other conditions constant. Computations of these curves were performed using the second-order approximation for the mean bubble radius and the first-order ap-
proximation for the amplitude of oscillation. Substantially
different signatures of the bubble dynamics created by dif-
ferent amplitudes of oscillation, growth and collapse rates,
thresholds, and stable average radii can be used for evalua-
tion of $\beta$.

Figures 13–17 demonstrate the influence of initial con-
ditions on the vapor bubble dynamics in acoustic fields. In all
computed cases computations were performed using the
third-order approximation for the mean radius and the first-
order approximation for the oscillatory part. The second-
order approximation (67) provided the same results as the
third-order approximation (83) where $\Phi_{200} = 0$ was assumed.
Condition $\Phi_{200} = 0$ as well as the second-order approxima-
tion corresponds to the quasisteady initial temperature pro-
files in the liquid. If the bubble is placed into the liquid of
uniform temperature several initial bubble pulsations cause
increase or decrease of the averaged bubble wall temperature
which is treated by the present theory as a jump at $t_2 = 0$.
Condition $\Phi_{200} \neq 0$ describes this jump and can substantially
influence rectified heat transfer. Due to the initial jump of the
nonoscillatory component of temperature the mean bubble
size starts to grow or decay proportionally $A_t$. This is deter-
mined by the sign of the function $\Phi_{20}(a_0)$ which can be
negative or positive depending on $a_0, \beta, \omega, \text{ and other pa}-
rameters (see Fig. 13).

A comparison between computations using two different
initial conditions $\Phi_{200} = 0$ and $\Phi_{200} = \Phi_{20}(a_0)$ is shown in
Figs. 14–17. Curves in Fig. 14 were computed for the case
considered by Hao and Prosperetti. Due to the relatively low frequency phase transformations occur here in quasi-equilibrium conditions and computations were performed using the equilibrium scheme. The comparison between the present theory and straightforward numerical simulations shows a satisfactory agreement between the present theory with \( F_{\Phi_{200}} \) and numerical results. In the case \( F_{\Phi_{200}} = 0 \) much slower growth rates are realized. In reality there exist uncertainty in initial temperature field around the bubble. A bubble of 50 or 100 \( \mu m \) in radius cannot be created immediately in the bulk of uniformly heated liquid. It appears because of growth, collapse, coalescence or fragmentation of smaller or larger bubbles. The plotted results present two limiting cases when the time of bubble creation is very short or very large if compared with the time of establishing of quasisteady temperature profile around the bubble.

with \( F_{\Phi_{200}} = \Phi_{200}(a_{in}) \) and numerical results. In the case \( F_{\Phi_{200}} = 0 \) much slower growth rates are realized. In reality there exist uncertainty in initial temperature field around the bubble. A bubble of 50 or 100 \( \mu m \) in radius cannot be created immediately in the bulk of uniformly heated liquid. It appears because of growth, collapse, coalescence or fragmentation of smaller or larger bubbles. The plotted results present two limiting cases when the time of bubble creation is very short or very large if compared with the time of establishing of quasisteady temperature profile around the bubble.

FIG. 14. Comparison between computations with the initial quasi-steady temperature profile in the liquid \( (F_{\Phi_{200}} = 0, \text{the lower thick curve}) \), initial temperature jump \( (\Phi_{200} = \Phi_{200}(a_{in}), \text{the upper thick curve}) \), and the numerical results of Hao and Prosperetti (1998) (the thin dashed line). The bubble radius is normalized to the primary resonance radius \( a_r = 2.71 \text{ mm} \). The initial mean radius in computations using the present theory is \( a_{in} = 0.1 \text{ mm} \).

FIG. 15. Comparison between computations using the third-order asymptotic theory with the initial temperature jump \( (\Phi_{\Phi_{200}} = \Phi_{200}(a_{in}), a_{in} = 98 \text{ \( \mu m \)} \), the solid curves) and straightforward numerical simulations with the initial bubble radius 100 \( \mu m \) using the detailed equations and a finite-difference scheme (the gray region and the dashed curve). Letters \( L, M, \) and \( U \) near the curves relates to the lower, mean, and upper slowly changing bubble radius. The curves \( L \) and \( U \) were computed by addition and substraction of the amplitude of bubble oscillation predicted by the linear theory.

FIG. 16. Dynamics of the mean vapor bubble radius in water with different initial temperature profiles near the bubble, \( F_{\Phi_{200}} \), and different initial radii, \( a_{in} \), shown near the curves. The thick lines correspond to the temperature jump at \( t=0 \) \( (\Phi_{\Phi_{200}} = \Phi_{200}(a_{in})) \) while the thin lines correspond to the initial quasisteady temperature profiles near the bubble \( (\Phi_{\Phi_{200}} = 0) \).

FIG. 17. As in Fig. 15 for subcooled liquid helium.
Figure 15 demonstrates a comparison between the present asymptotic theory and straightforward numerical computations based on the detailed governing equations. The straightforward computations were performed using an explicit finite-difference scheme with a nonuniform grid enabling spatial resolution of thin thermal boundary layers in the liquid and in the vapor. The accuracy of the scheme was tested by comparison with small amplitude asymptotic theory and with simulations of Hao and Prosperetti. In the straightforward computations the time step required for stability and accuracy was very small. This resulted in a huge difference in the CPU time required by the two approaches (for the illustrated case the asymptotic solution was $10^4$ times faster than the straightforward solution). It is seen that the asymptotic solution combining the third-order approximation for the mean bubble radius and the first-order approximation for the phase and amplitude of fast bubble radius oscillation is consistent with the straightforward computations. We mentioned above how the residual term of the asymptotics (83) can be secular or nonsecular. Presumably, in the computed case it is nonsecular. Otherwise the asymptotic theory should show a discrepancy at times $\omega t/2\pi \sim e^{-\pi^2/2} \sim 100$. However, the comparison shows that even at maximum times $\omega t/2\pi \sim 10^3 - 10^4$, which could be achieved with the straightforward computations, the present asymptotic theory predicts the bubble dynamics well.

Computations presented in Fig. 16 show nonmonotonic curves for the mean bubble radius for the case $\Phi_{200} = \Phi_{20}(a_m)$ while for the case $\Phi_{200} = 0$ which coincides with the second-order approximation computations with the same initial radius, a monotonic increase or decrease of the mean bubble size was obtained. It is also noteworthy that the time required to achieve the stable bubble size ($a_m = 71 \mu m$) in the illustrated case substantially differ for the two different initial temperature profiles in the liquid. However, the time required to reach the state independent of initial bubble size is approximately the same for different initial temperature profiles (of order $\omega t/2\pi \sim 10^3$). We found that after reaching this state efficient computations for $\Phi_{200} = 0$ can be performed using large-time asymptotic expression (89). Figure 16 demonstrates that the stable bubble size can be achieved for very large times ($10^7$ cycles, or about 3 minutes). However, we cannot prove this result (due to unknown structure of the residual terms), and we show it because of a good agreement with straightforward computations at large times (see Fig. 15). Note that Hao and Prosperetti reported that the bubble radius did not show stabilization at $\omega t/2\pi \sim 10^3$.

The possibility for the existence of steady bubble oscillations near the low-frequency resonance was discussed in Refs. 25 and 14 and suggested as an explanation of the observed small stable bubbles in liquid helium. The present theory does not predict stable bubbles at 4.2 K and atmospheric pressure for 52 kHz acoustic field in the range of 15 $\mu m$. However computations showed that at the parameters of experiment the unstable threshold radius, $a_{th}$, of order 15 $\mu m$ can be realized. Although this radius is unstable, the times of the instability development can reach several seconds, or hundreds of thousands of periods of oscillation [see Fig. 17; the thick and thin lines correspond to $\Phi_{200} = \Phi_{20}(a_m)$ and $\Phi_{200} = 0$, respectively] which is in the same order of magnitude as the times observed in experiments. In the experimental photographs some patterns of bubbles in standing waves were seen. Pattern formation in the liquid with vapor bubbles is a result of strong acoustic field/bubble interaction leading to self-organization of the bubbles. Such interaction as well as the bubble drift to the nodes or antinodes of the standing wave can also provide stabilization of bubbles of a certain size.

VIII. CONCLUSIONS

In the present study a multiscale asymptotic technique applicable to investigation of the effect of rectified heat transfer was extended to include effects of surface tension, viscosity, acoustic radiation, and initial temperature jump. The surface tension heavily influences the slow time vapor bubble dynamics in high frequency acoustic fields. The condensation–evaporation–capillary (low-frequency, or second) resonance of vapor bubbles at moderate values of the accommodation coefficient is responsible for the threshold of acoustic vapor cavitation and possible stabilization of bubbles with sizes essentially smaller than the primary resonance size.

Effects of nonequilibrium phase transitions substantially influence the dynamics of vapor bubbles in water at ambient atmospheric pressure for acoustic frequencies of order 10 kHz and higher. Depending on the contamination of the system and corresponding values of the accommodation coefficient the behavior and resulting acoustic signatures of vapor bubble dynamics including the amplitude and phase of oscillations, the rate of growth or collapse, and the values of the threshold and stable radii can vary significantly. The number of threshold and stable radii of vapor bubbles in an acoustic field was also found to depend on the value accommodation coefficient. Such dependences are complex and in many cases nonmonotonic due to a combination of several effects connected with the phase shifts in oscillations of bubble radius and thermodynamic parameters, the intensity of the condensation/evaporation, and the degree of superheating or subcooling of the liquid.

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