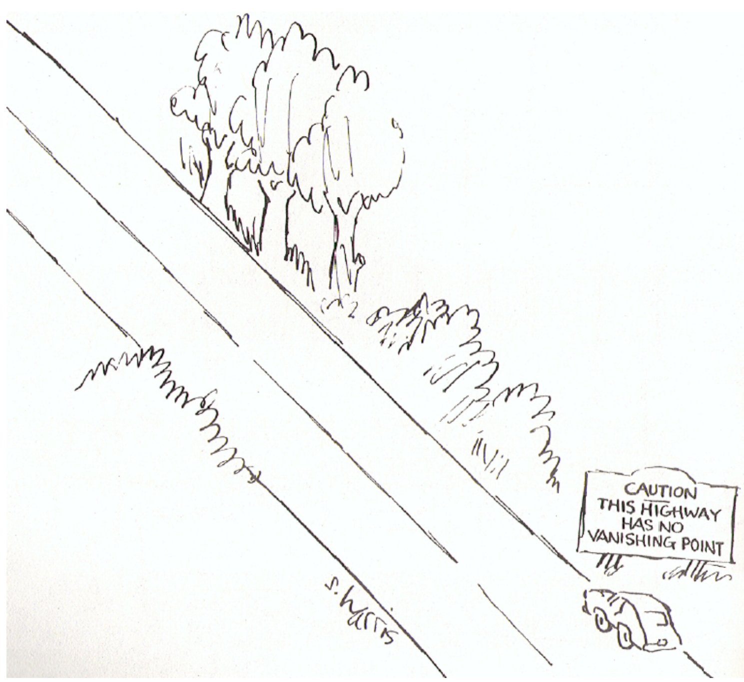
# Lecture 4
# Mobot Kinematics and Control

## CSE390/MEAM420-520

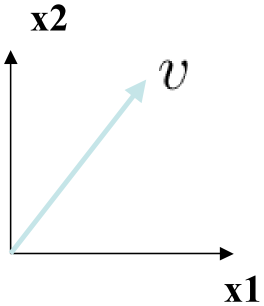
Some notes taken from Siegwart&Nourbakhsh

**Review: Algebra and Geometry**



# Vectors

- Ordered set of numbers: (1,2,3,4)
- Example: (*x*,*y*,*z*) coordinates of pt in space.




$$v = (x_1, x_2, \ldots, x_n)$$

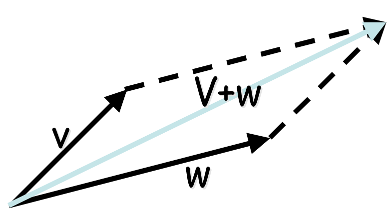
$$\|v\| = \sqrt{\sum_{i=1}^{n} x_i^2}$$

If $\|v\| = 1$, $v$ is a unit vector

**Distance = norm**

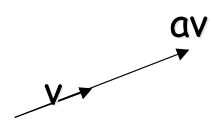
# Vector Addition

$$\mathbf{v} + \mathbf{w} = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$

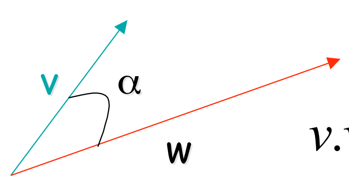


# Scalar Product

$$a\mathbf{v} = a(x_1, x_2) = (ax_1, ax_2)$$




# Inner (dot) Product

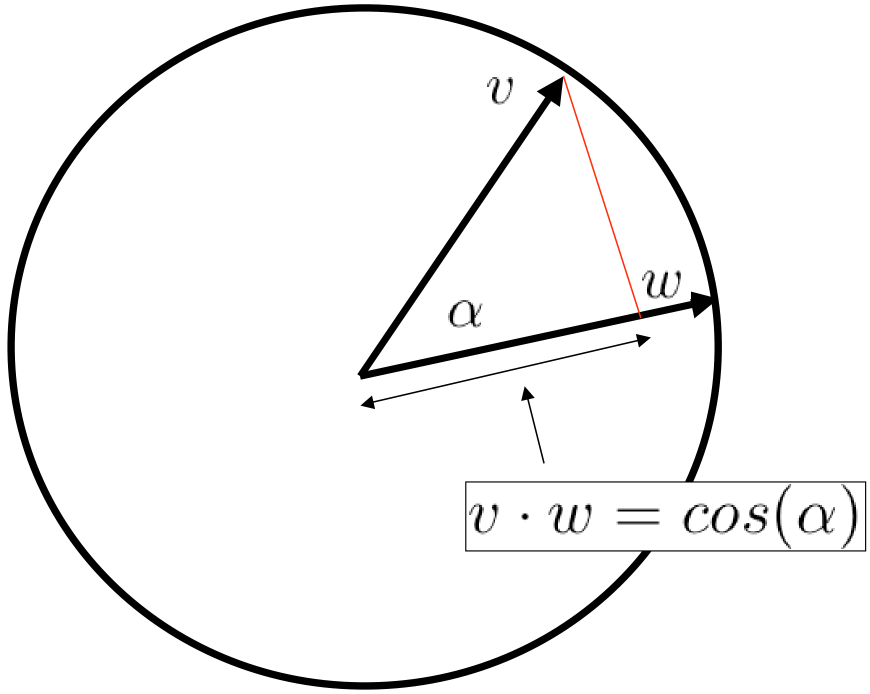



$$v.w = (x_1, x_2).(y_1, y_2) = x_1 y_1 + x_2.y_2$$

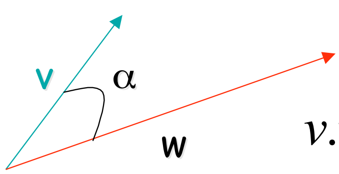
The inner product is a SCALAR!

**Angle:**

# Unit circle, angle




# Inner (dot) Product




$$v.w = (x_1, x_2).(y_1, y_2) = x_1 y_1 + x_2.y_2$$

**Angle:** $$v.w = (x_1, x_2).(y_1, y_2) = \| v \| \cdot \| w \| \cos\alpha$$

$$v.w = 0 \Leftrightarrow v \perp w$$

< = > V,W are independent of each other

# Matrices

$$A_{n\times m} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ a_{31} & a_{32} & \cdots & a_{3m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

Sum:

$$C_{n\times m} = A_{n\times m} + B_{n\times m}$$

$$c_{ij} = a_{ij} + b_{ij}$$

A and B must have the same dimensions

# Matrices

Product:

$$C_{n\times p} = A_{n\times m} B_{m\times p}$$

A and B must have compatible dimensions

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

$$A_{n\times n} B_{n\times n} \neq B_{n\times n} A_{n\times n}$$

**Is matrix multiplication associative?**

# Matrices

Product:

$$C_{n\times p} = A_{n\times m} B_{m\times p}$$

A and B must have compatible dimensions

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

$$A_{n\times n} B_{n\times n} \neq B_{n\times n} A_{n\times n}$$

Identity Matrix:

$$I = \begin{pmatrix} 1 & 0 & \text{O} & 0 \\ 0 & 1 & \text{O} & 0 \\ \text{O} & \text{O} & \text{O} & \text{O} \\ 0 & 0 & \text{O} & 1 \end{pmatrix} \qquad IA = AI = A$$

# Matrices

Transpose:

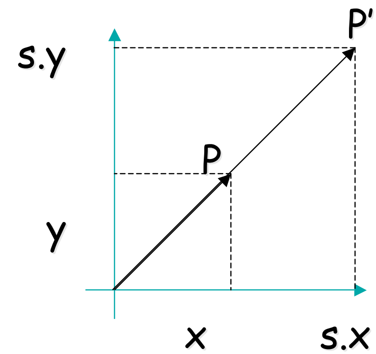
$$C_{m\times n} = A^T{}_{n\times m} \qquad (A+B)^T = A^T + B^T$$

$$c_{ij} = a_{ji} \qquad (AB)^T = B^T A^T$$

If $A^T = A$ A is symmetric

**Dot product:** $v \cdot w = vw^T$

# Scaling Equation




$$\mathbf{P} = (x, y)$$

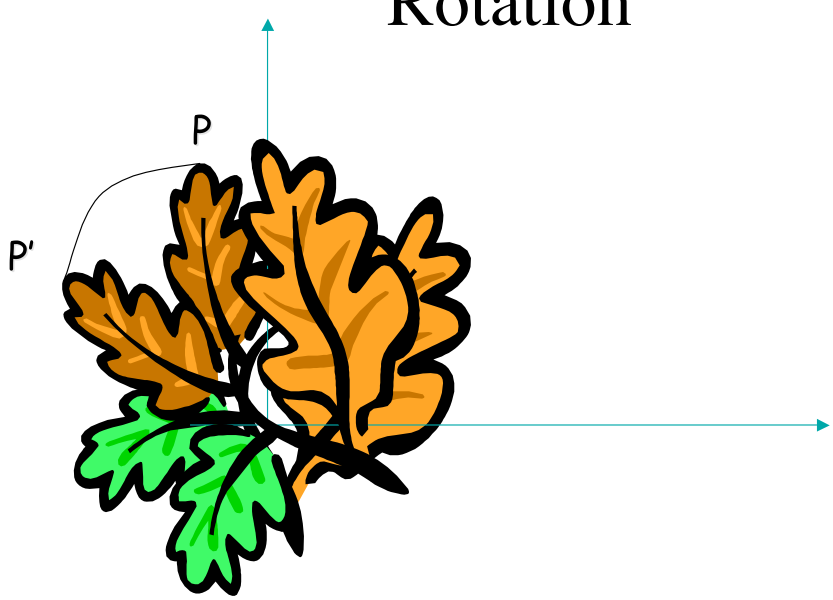
$$\mathbf{P}' = (sx, sy)$$

$$\mathbf{P}' = s \cdot \mathbf{P}$$

$$\mathbf{P}' \rightarrow \begin{bmatrix} sx \\ sy \end{bmatrix} = \underbrace{\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}}_{\mathbf{S}} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

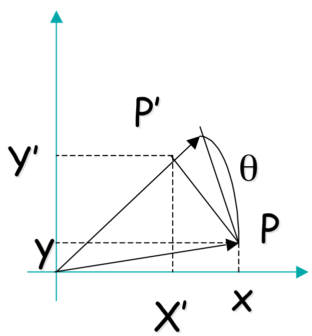
$$\mathbf{P}' = \mathbf{S} \cdot \mathbf{P}$$

# Rotation



# Rotation Equations

Counter-clockwise rotation by an angle $\theta$




$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{P'} = \mathbf{R.P}$$

# Degrees of Freedom

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

R is 2x2 $\Rightarrow$ 4 elements

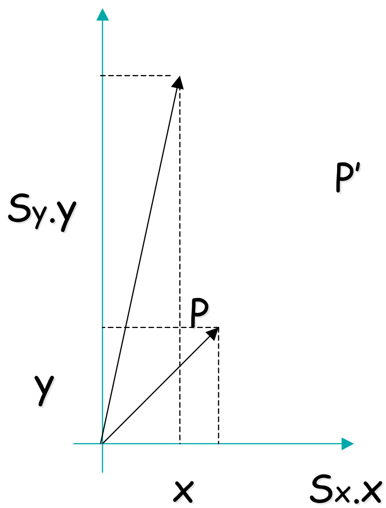
BUT! There is only 1 degree of freedom: $\theta$

The 4 elements must satisfy the following constraints:

$$\mathbf{R} \cdot \mathbf{R}^{\mathrm{T}} = \mathbf{R}^{\mathrm{T}} \cdot \mathbf{R} = \mathbf{I}$$

$$\det(\mathbf{R}) = 1$$

# Stretching Equation




$$\mathbf{P} = (x, y)$$

$$\mathbf{P'} = (s_x x, s_y y)$$

$$\mathbf{P'} \to \begin{bmatrix} s_x x \\ s_y y \end{bmatrix} = \underbrace{\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}}_{\mathbf{S}} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{P'} = \mathbf{S} \cdot \mathbf{P}$$

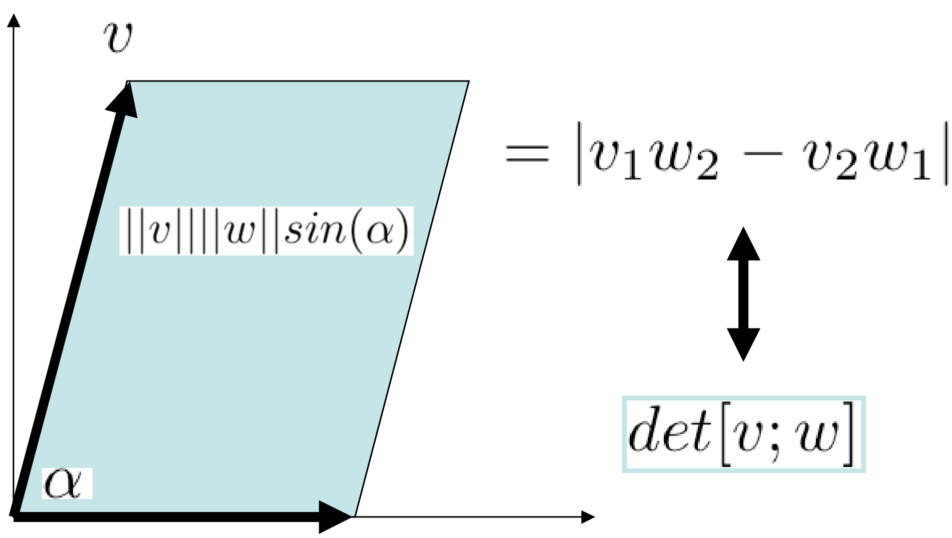
# Matrices

Determinant: A must be square

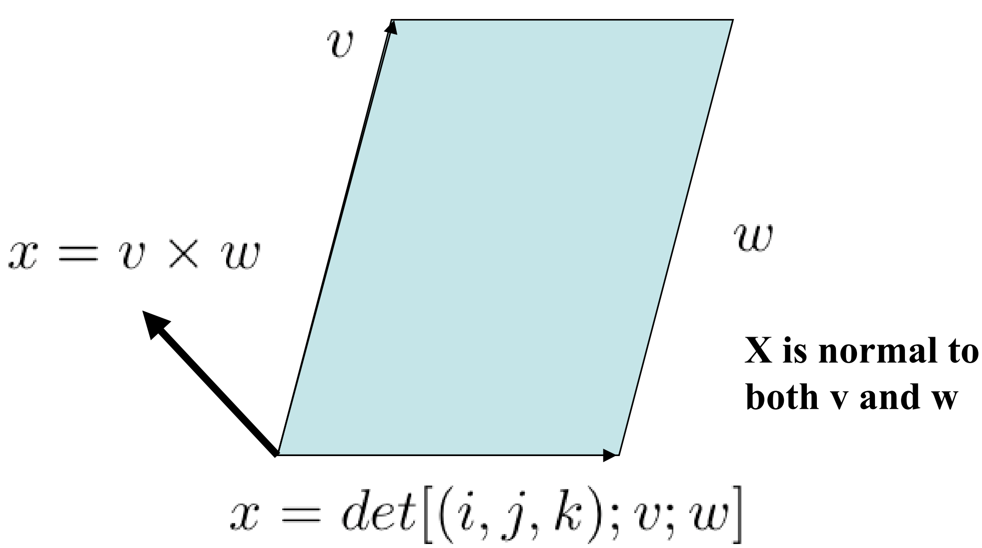
$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

# Volume = determinant




# Cross product



# Matrices

Inverse:

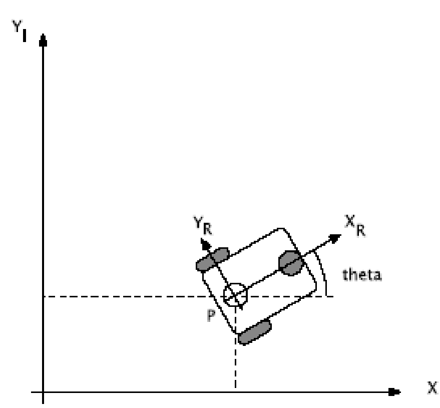
A must be square

$$A_{n\times n}A^{-1}{}_{n\times n} = A^{-1}{}_{n\times n}A_{n\times n} = I$$

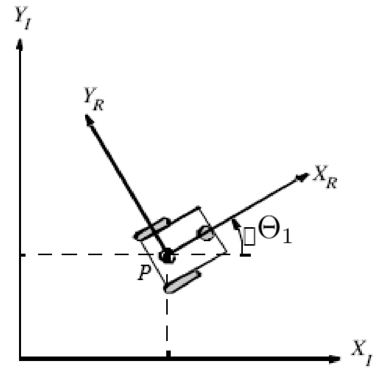
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

- Inertial reference frame (I)
- Robot references frame (R)
- Robot pose

$$\xi_I = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$



robot motion: $\dot{\xi}_I = [\dot{x}, \dot{y}, \dot{\theta}]^T$;

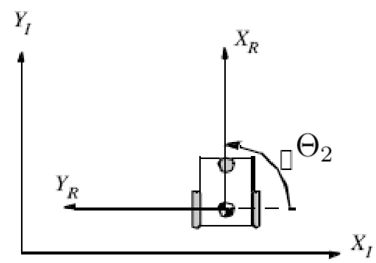



- The relation between the references frame is through the standard orthogonal rotation transformation:

$$R(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\theta = \theta_2 - \theta_1$$

$$\dot{\xi_R} = R(\theta)\dot{\xi_I}$$






- Forward kinematics

$$\dot{\xi} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = f(\dot{\varphi}_1, \ldots, \dot{\varphi}_n, \beta_1, \ldots, \beta_m, \dot{\beta}_1, \ldots, \dot{\beta}_m)$$

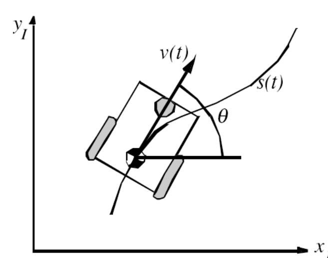
- Inverse kinematics

$$\begin{bmatrix} \dot{\varphi}_1 & \ldots & \dot{\varphi}_n & \beta_1 & \ldots & \beta_m & \dot{\beta}_1 & \ldots & \dot{\beta}_m \end{bmatrix}^T = f(\dot{x}, \dot{y}, \dot{\theta})$$

- Why not

$$\begin{bmatrix} x \\ y \\ \theta \end{bmatrix} = f(\varphi_1, \ldots, \varphi_n, \beta_1, \ldots, \beta_m)$$

the relation is not straight forward. See later.

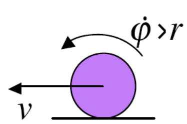


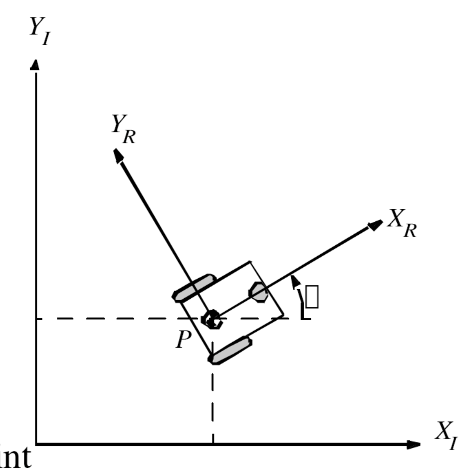
**Kinematics Models of wheel**
(rolling and sliding contraints)

**Mobil robot(mobot) maneuverability**
(possible space and velocity it can reach)

**Mobot Kinematic Control**
(Sequence of movements toward target)

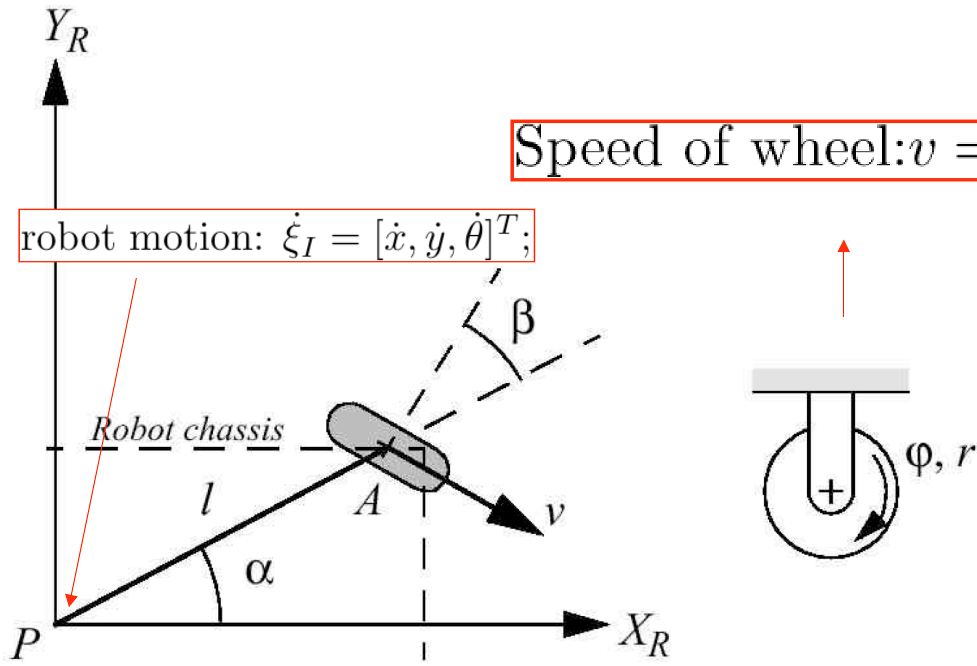
- Movement on a horizontal plane
- Point contact of the wheels
- Wheels not deformable
- Pure rolling
  - v = *0 at contact point*
- No slipping, skidding or sliding
- No friction for rotation around contact point
- Steering axes orthogonal to the surface
- Wheels connected by rigid frame (chassis)



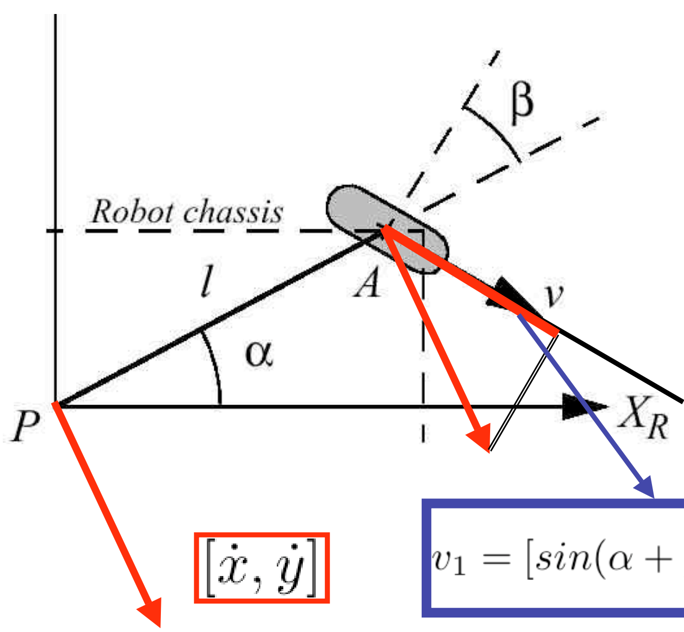
Speed of wheel: $v = r\dot{\varphi}$

robot motion: $\dot{\xi}_I = [\dot{x}, \dot{y}, \dot{\theta}]^T$;




robot motion: $\dot{\xi}_I = [\dot{x}, \dot{y}, \dot{\theta}]^T$;

Speed of wheel: $v = r\dot{\varphi}$

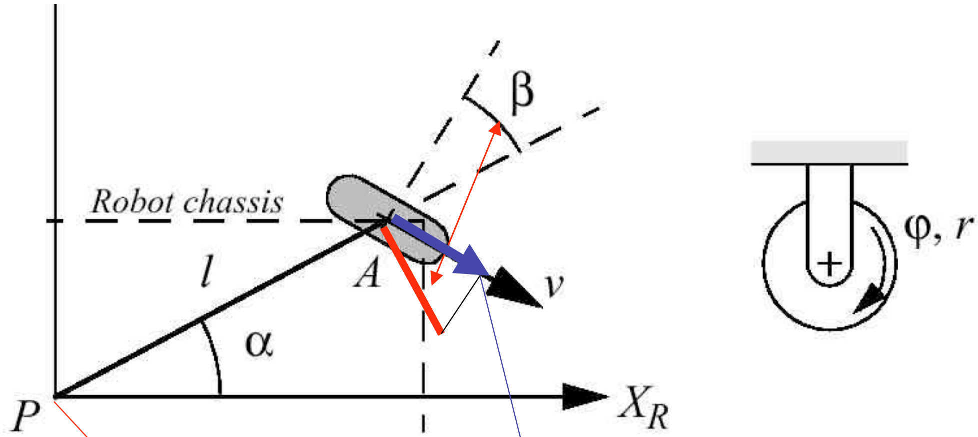



$[\dot{x}, \dot{y}]$

$$v_1 = [sin(\alpha + \beta], -cos(\alpha + \beta] \cdot \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

robot motion: $\dot{\xi}_I = [\dot{x}, \dot{y}, \dot{\theta}]^T$;

Speed of wheel: $v = r\dot{\varphi}$




Rotation:

$\boxed{\dot{\theta}}$

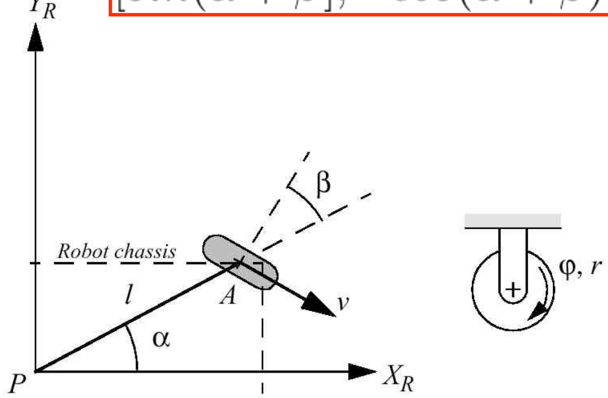
$$\boxed{v_{rotate} = -l cos(\beta)\dot{\theta}}$$

robot motion: $\dot{\xi}_I = [\dot{x}, \dot{y}, \dot{\theta}]^T$;

Speed of wheel: $v = r\dot{\varphi}$

$$\boxed{v = r\dot{\varphi} = [sin(\alpha + \beta]\dot{x}, -cos(\alpha + \beta)\dot{y}, -lcos(\beta)\dot{\theta}]}$$

$$\boxed{[sin(\alpha + \beta], -cos(\alpha + \beta), -lcos(\beta)] \cdot \dot{\xi}_I - r\dot{\varphi} = 0}$$



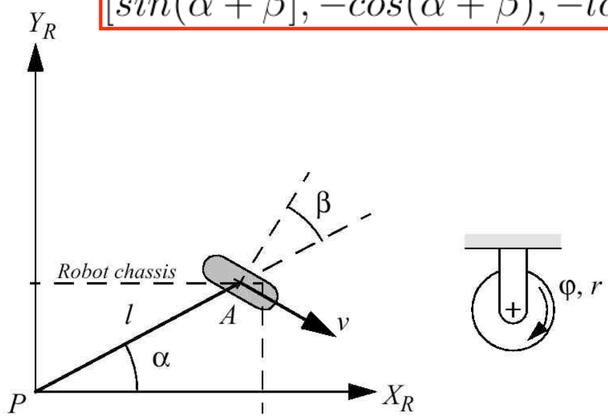
robot motion: $\dot{\xi}_I = [\dot{x}, \dot{y}, \dot{\theta}]^T$;

Speed of wheel: $v = r\dot{\varphi}$

$$v = r\dot{\varphi} = [sin(\alpha + \beta]\dot{x}, -cos(\alpha + \beta)\dot{y}, -lcos(\beta)\dot{\theta}]$$

$$[sin(\alpha + \beta], -cos(\alpha + \beta), -lcos(\beta)] \cdot R(\theta_R)\dot{\xi}_I - r\dot{\varphi} = 0$$

Transformation to the Reference frame




Motion caused by wheel motion, rolling constraints:

$$[sin(\alpha + \beta], -cos(\alpha + \beta), -lcos(\beta)] \cdot R(\theta_R)\dot{\xi}_I - r\dot{\varphi} = 0$$

Motion in the orthogonal plane must be 0, sliding constraints:

$$[cos(\alpha + \beta) \;\; sin(\alpha + \beta) \;\; l \cdot sin(\beta)]R(\theta_R)\dot{\xi}_I = 0$$

Kinematics Models of wheel
(rolling and sliding contraints)

↓

Mobil robot(mobot) maneuverability
(possible space and velocity it can reach)

↓

Mobot Kinematic Control
(Sequence of movements toward target)

Suppose we have a total of $N=N_f+N_s$ standard wheels

- *We can develop the equations for the constraints in matrix forms :*
- *Rolling*

$$J_1(\beta_s)R(\theta)\dot{\xi}_I + J_2\dot{\varphi} = 0 \qquad \varphi(t) = \begin{bmatrix} \varphi_f(t) \\ \varphi_s(t) \end{bmatrix}$$

$$J_1(\beta_s) = \begin{bmatrix} J_{1f} \\ J_{1s}(\beta_s) \end{bmatrix}_{(N_f+N_s)\times 3} = diag(r_1 \cdots r_N)$$

Suppose we have a total of $N=N_f+N_s$ standard wheels

- *We can develop the equations for the constraints in matrix forms :*
- *Rolling*

$$J_1(\beta_s)R(\theta)\dot{\xi}_I + J_2\dot{\varphi} = 0 \qquad \varphi(t) = \begin{bmatrix} \varphi_f(t) \\ \varphi_s(t) \end{bmatrix}$$

- *Lateral movement*

$$C_1(\beta_s)R(\theta)\dot{\xi}_I = 0 \qquad C_1(\beta_s) = \begin{bmatrix} C_{1f} \\ C_{1s}(\beta_s) \end{bmatrix}_{(N_f+N_s)\times 3}$$

Examples: differential drive + omidirectional drive

## Mobile Robot Maneuverability: Degree of Mobility

- To avoid any lateral slip the motion vector $R(\theta)\dot{\xi}_I$ has to satisfy the following constraints:

$$C_{1f} R(\theta)\dot{\xi}_I = 0$$

$$C_{1s}(\beta_s) R(\theta)\dot{\xi}_I = 0$$

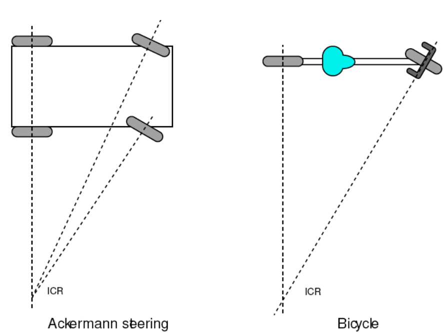
$$C_1(\beta_s) = \begin{bmatrix} C_{1f} \\ C_{1s}(\beta_s) \end{bmatrix}$$

- Mathematically:
  - $R(\theta)\dot{\xi}_I$ *must belong to the* null space *of the projection matrix* $C_1(\beta_s)$
  - Null space *of* $C_1(\beta_s)$ *is the space* N *such that for any vector* n *in* N

$$C_1(\beta_s) \cdot n = 0$$

## Mobile Robot Maneuverability: More on Degree of Mobility

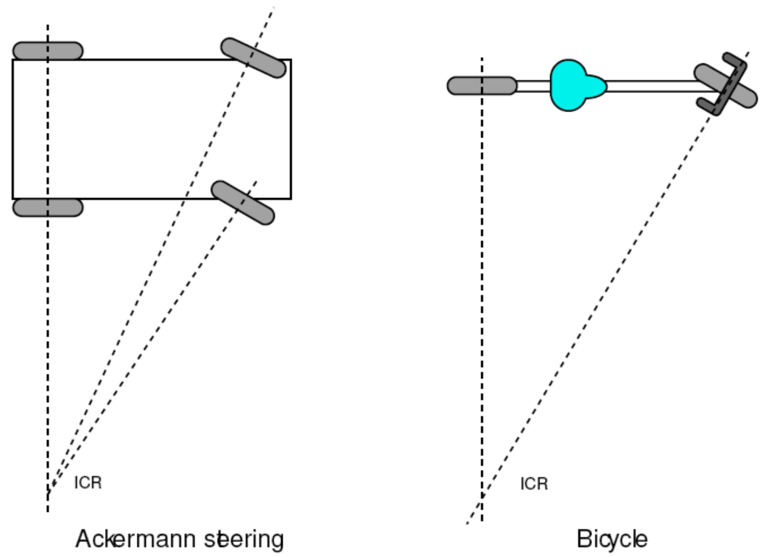
- Robot chassis kinematics is a function of the set of *independent constraints* $rank\left[C_1(\beta_s)\right]$
  - *the greater the rank of ,* $C_1(\beta_s)$ *the more constrained is the mobility*




The $rank(C_1)$ defines the number of independent constraints

ICR is the Null space of the C1

- *Geometrically this can be shown by the* Instantaneous Center of Rotation (ICR)




ICR is the Null space of the $C_1$; $Null(C_1) + Rank(C_1) = 3$

The degree of mobility is defined by the dimensionality of the null space of $C_1$ which for a mobile platform is equal to:

$$\delta_m = dim(null(C_1)) = 3 - rank(C_1)$$

The degree of mobility is defined by the dimensionality of the null space of $C_1$ which for a mobile platform is equal to:

$$\delta_m = dim(null(C_1)) = 3 - rank(C_1)$$

Examples:

| Robot | $\delta_m$ |
|---|---|
| Differential drive | 2 |
| Bicycle | 1 |

- Steerability is the number of independent DOF that can be controlled

$$\delta_s = rank(C_{1s})$$

- Similarly the degree of maneuverability is defined as

$$\delta_M = \delta_m + \delta_s$$

- Degree of Maneuverability

$$\delta_M = \delta_m + \delta_s$$

  - *Two robots with same* $\delta_M$ *are not necessary equal*
  - *Example: Differential drive and Tricycle (next slide)*

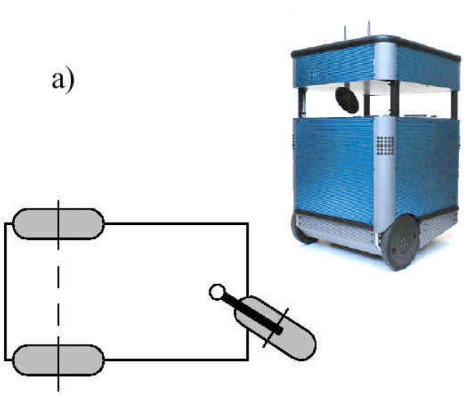
  - *For any robot with* $\delta_M = 2$ *the ICR is always constrained to* lie on a line
  - *For any robot with* $\delta_M = 3$ *the ICR is not constrained an can* be set to any point on the plane

- The Synchro Drive example: $\delta_M = \delta_m + \delta_s = 1 + 1 = 2$

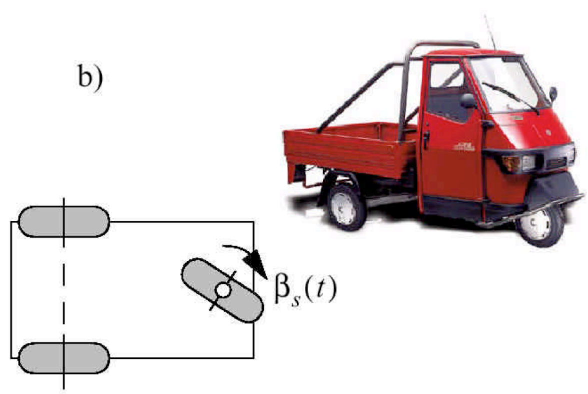
## Mobile Robot Maneuverability: Wheel Configurations

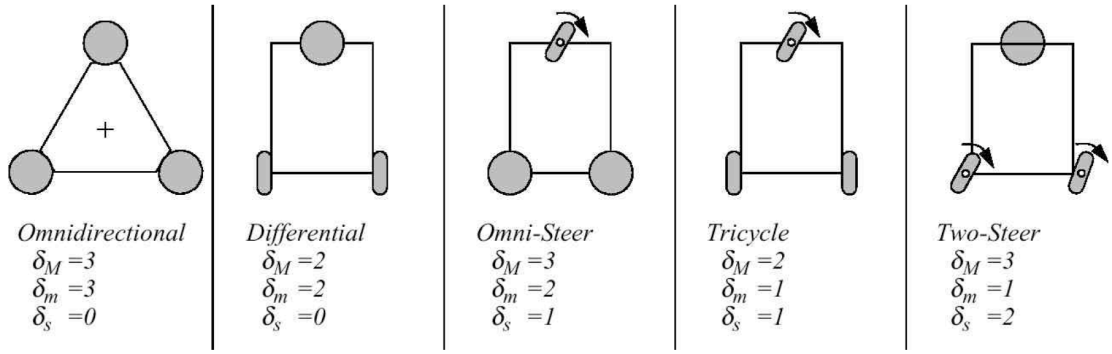
- Differential Drive Tricycle

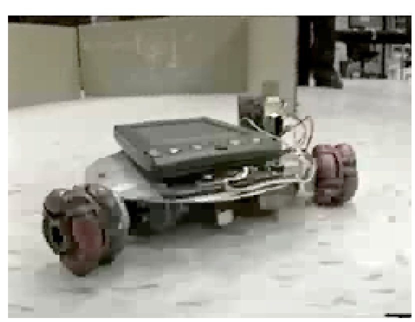
a)



b)





Palm Pilot Robot, CMU

Kinematics Models of wheel

(rolling and sliding contraints)

↓

Mobil robot(mobot) maneuverability

(possible space and velocity it can reach)

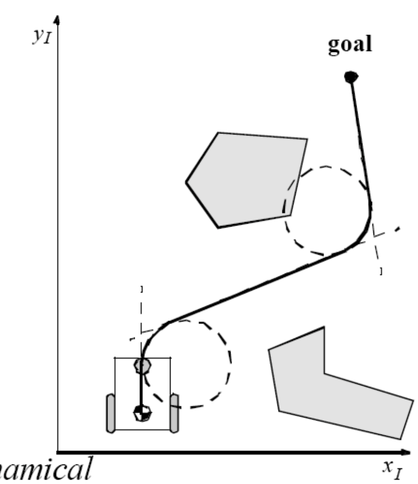
↓

Mobot Kinematic Control

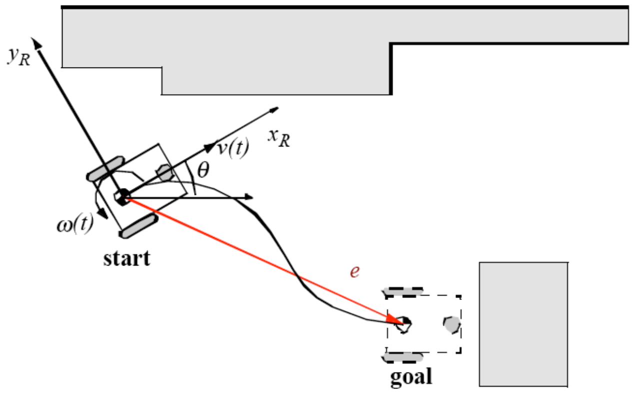
(Sequence of movements toward target)

## Motion Control: Open Loop Control

- trajectory (path) divided in motion segments of clearly defined shape:
  - *straight lines and segments of a circle.*
- control problem:
  - *pre-compute a smooth trajectory based on line and circle segments*
- Disadvantages:
  - *It is not at all an easy task to pre-compute a feasible trajectory*
  - *limitations and constraints of the robots velocities and accelerations*
  - *does not adapt or correct the trajectory if dynamical changes of the environment occur.*
  - *The resulting trajectories are usually not smooth*



## Motion Control: Feedback Control, Problem Statement




- Find a control matrix $K$, if exists

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \end{bmatrix}$$

with $k_{ij}$=$k(t,e)$

- such that the control of $v(t)$ and $\omega(t)$

$$\begin{bmatrix} v(t) \\ \omega(t) \end{bmatrix} = K \cdot e = K \cdot {}^{R}\begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

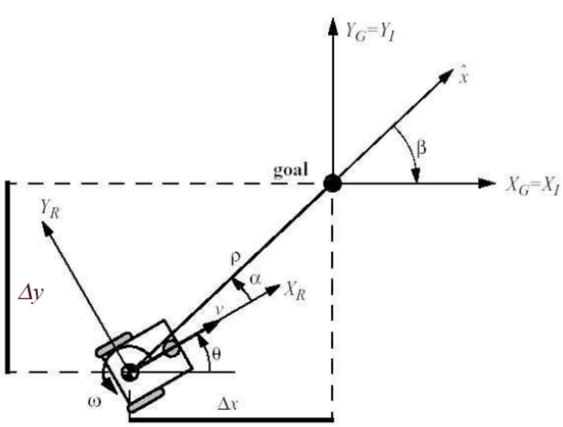
- drives the error e to zero.

$$\lim_{t \to \infty} e(t) = 0$$



Motion Control:

## Kinematic Position Control

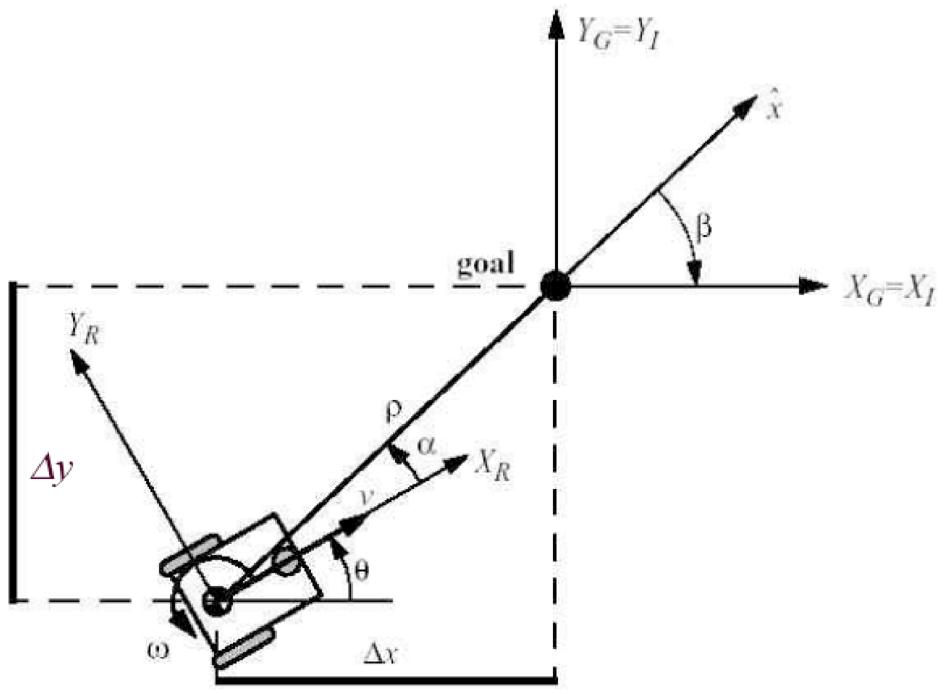



The kinematic of a differential drive mobile robot described in the initial frame $\{x_I, y_I, \theta\}$ is given by,

$${}^{I}\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}$$

where and are the linear velocities in the direction of the $x_I$ and $y_I$ of the initial frame.

Let $\alpha$ denote the angle between the $x_R$ axis of the robots reference frame and the vector connecting the center of the axle of the wheels with the final position.

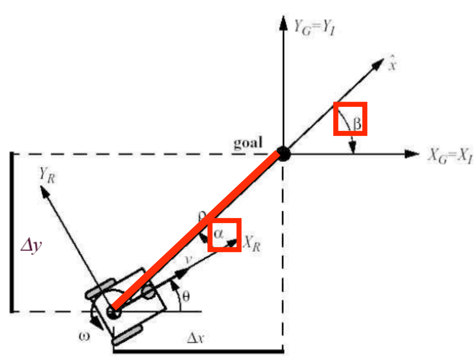



**Coordinates transformation into polar coordinates with its origin at goal position:**

$$\rho = \sqrt{\Delta x^2 + \Delta y^2}$$

$$\alpha = -\theta + \operatorname{atan2}(\Delta y, \Delta x)$$

$$\beta = -\theta - \alpha$$

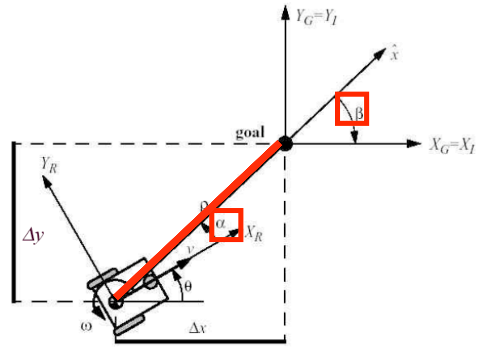


**Coordinates transformation into polar coordinates with its origin at goal position:**

$$\rho = \sqrt{\Delta x^2 + \Delta y^2}$$

$$\alpha = -\theta + \operatorname{atan2}(\Delta y, \Delta x)$$

$$\beta = -\theta - \alpha$$




**System description, in the new polar coordinates**

$$\begin{bmatrix} \dot{\rho} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} -\cos\alpha & 0 \\ \dfrac{\sin\alpha}{\rho} & -1 \\ -\dfrac{\sin\alpha}{\rho} & 0 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix} \qquad \begin{bmatrix} \dot{\rho} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} \cos\alpha & 0 \\ -\dfrac{\sin\alpha}{\rho} & 1 \\ \dfrac{\sin\alpha}{\rho} & 0 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}$$

*for* $I_1 = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$ *for* $I_2 = (-\pi, -\pi/2] \cup (\pi/2, \pi]$



# Kinematic Position Control: Remarks

- The coordinates transformation is not defined at $x = y = 0$; as in such a point the determinant of the Jacobian matrix of the transformation is not defined, i.e. it is unbounded

- For $\alpha \in I_1$ the forward direction of the robot points toward the goal, for $\alpha \in I_2$ it is the backward direction.

- By properly defining the forward direction of the robot at its initial configuration, it is always possible to have $\alpha \in I_1$ at $t$=0. However this does not mean that $\alpha$ remains in $I_1$ for all time $t$.

## Kinematic Position Control: The Control Law

- It can be shown, that with

$$v = k_\rho \rho \qquad \omega = k_\alpha \alpha + k_\beta \beta$$

$$\begin{bmatrix} \dot{\rho} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} -\cos\alpha & 0 \\ \frac{\sin\alpha}{\rho} & -1 \\ -\frac{\sin\alpha}{\rho} & 0 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}$$

$$\begin{bmatrix} \dot{\rho} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} -k_\rho \rho \cos\alpha \\ k_\rho \sin\alpha - k_\alpha \alpha - k_\beta \beta \\ -k_\rho \sin\alpha \end{bmatrix}$$

## Kinematic Position Control: The Control Law

- It can be shown, that with

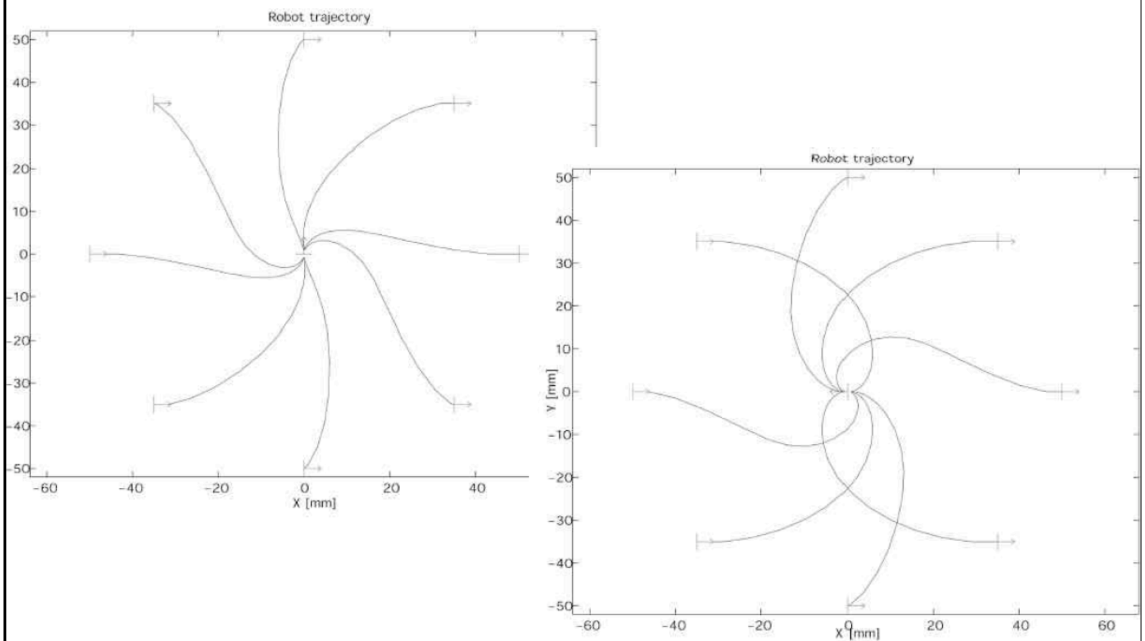
$$v = k_\rho \rho \qquad \omega = k_\alpha \alpha + k_\beta \beta$$

the feedback controlled system

$$\begin{bmatrix} \dot{\rho} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} -k_\rho \rho \cos\alpha \\ k_\rho \sin\alpha - k_\alpha \alpha - k_\beta \beta \\ -k_\rho \sin\alpha \end{bmatrix}$$

- will drive the robot to $(\rho, \alpha, \beta) = (0,0,0)$
- The control signal $v$ has always constant sign,
  - *the direction of movement is kept positive or negative during movement*
  - *parking maneuver is performed always in the most natural way and without ever inverting its motion.*

## Kinematic Position Control: Resulting Path

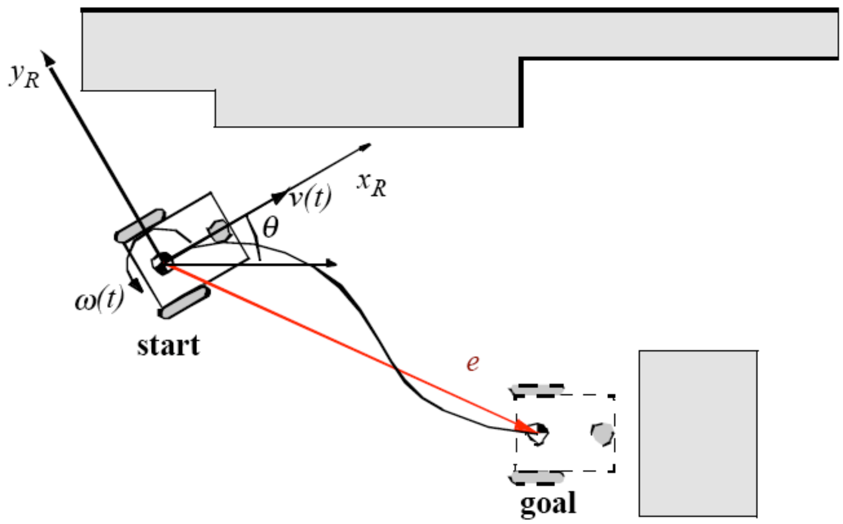



## Kinematic Position Control: Stability Issue

- It can further be shown, that the closed loop control system is locally exponentially stable if

$$k_\rho > 0 \;\; ; \;\; k_\beta < 0 \;\; ; \;\; k_\alpha - k_\rho > 0$$

Why there is a “S” curve shape to the path?




Kinematics Models of wheel
(rolling and sliding contraints)

Mobil robot(mobot) maneuverability
(possible space and velocity it can reach)

Optional Slides, will return to this in path planning lectures.

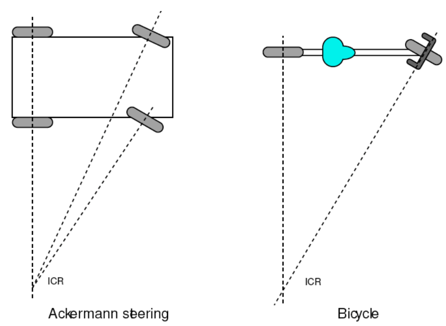
Mobot Kinematic Control
(Sequence of movements toward target)

# Degree of Freedom DOF

= possible space (x,y,theta) a robot can reach

What is the DOF of the Ackerman vehicle?

Is Degree of Freedom of robot same as its maneuverability?




# Differentiable Degree of Freedom DDOF

= possible velocity $(\dot{x}, \dot{y}, \dot{\theta})$ a robot can reach

DDOF = degree of Mobility, $\delta_m$

- *Bicycle:* $\delta_M = \delta_m + \delta_s = 1 + 1$ *DDOF = 1; DOF=3*

Omnibot: $\delta_M = \delta_m + \delta_s = 3 + 0 = 3;$

DDOF = 3; DOF = 3.

## Mobile Robot Workspace: Degrees of Freedom, Holonomy

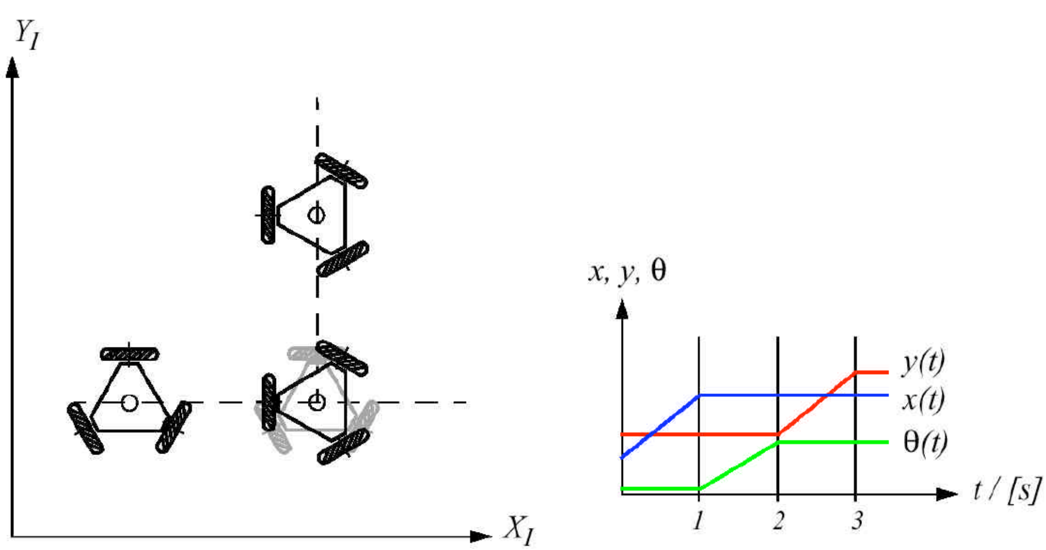
- DOF *degrees of freedom*:
  - *Robots ability to achieve various poses*
- DDOF *differentiable degrees of freedom*:
  - *Robots ability to achieve various path*

$$DDOF \leq \delta_m \leq DOF$$

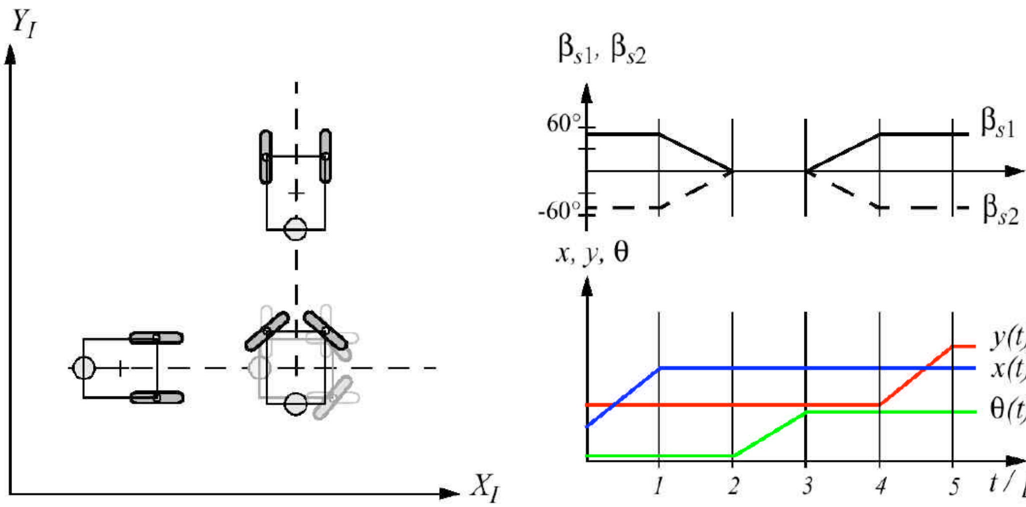
- Holonomic Robots
  - *A holonomic kinematic constraint can be expressed a an explicit function of position variables only*
  - *A non-holonomic constraint requires a different relationship, such as the derivative of a position variable*
  - Fixed and steered standard wheels impose non-holonomic constraints

**Robot is Holonomic <=> DDOF = DOF**

## Path / Trajectory Considerations: Omnidirectional Drive



## Path / Trajectory Considerations: Two -Steer

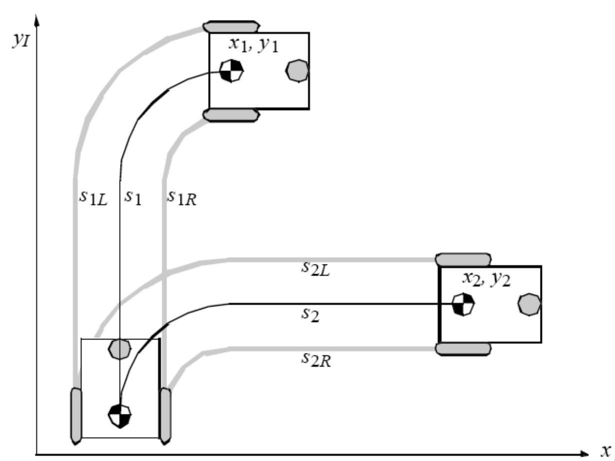





## Mobile Robot Kinematics: Non-Holonomic Systems

$s_1 = s_2$ ; $s_{1R} = s_{2R}$ ; $s_{1L} = s_{2L}$

but: $x_1 \neq x_2$ ; $y_1 \neq y_2$



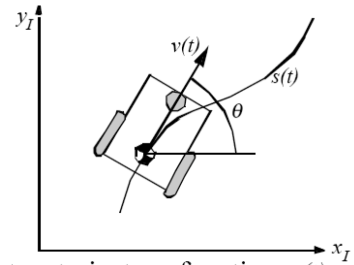

- Non-holonomic systems
  - *differential equations are not integrable to the final position.*
  - *the measure of the traveled distance of each wheel is not sufficient to calculate the final position of the robot. One has also to know how this movement was executed as a function of time.*

# Non-Holonomic Systems: Mathematical Interpretation

- A mobile robot is running along a trajectory $s(t)$.
  At every instant of the movement its velocity $v(t)$ is:

$$v(t) = \frac{\partial s}{\partial t} = \frac{\partial x}{\partial t}\cos\theta + \frac{\partial y}{\partial t}\sin\theta$$

$$ds = dx\cos\theta + dy\sin\theta$$




- Function $v(t)$ is said to be integrable (holonomic) if there exists a trajectory function $s(t)$ that can be described by the values $x$, $y$, and $\theta$ only.

$$s = s(x, y, \theta)$$

- This is the case if

$$\boxed{\frac{\partial^2 s}{\partial x \partial y} = \frac{\partial^2 s}{\partial y \partial x} \; ; \; \frac{\partial^2 s}{\partial x \partial \theta} = \frac{\partial^2 s}{\partial \theta \partial x} \; ; \; \frac{\partial^2 s}{\partial y \partial \theta} = \frac{\partial^2 s}{\partial \theta \partial y}}$$

Condition for integrable function

- With $s = s(x,y,\theta)$ we get for $ds$

$$ds = \frac{\partial s}{\partial x}dx + \frac{\partial s}{\partial y}dy + \frac{\partial s}{\partial \theta}d\theta$$