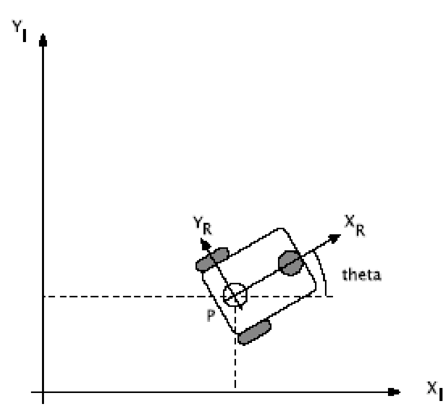
Lecture 3
Mobot Kinematics

CSE390/MEAM420-520

Some notes taken from Siegwart&Nourbakhsh

- Inertial reference frame (I)
- Robot references frame (R)
- Robot pose

$$\xi_I = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$



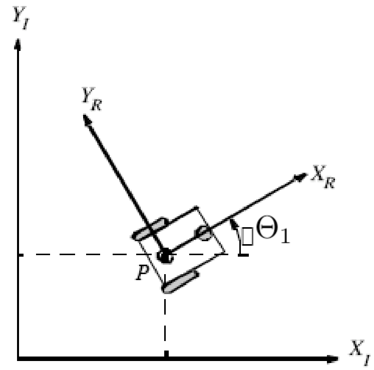
robot motion: $\dot{\xi}_I = [\dot{x}, \dot{y}, \dot{\theta}]^T$;

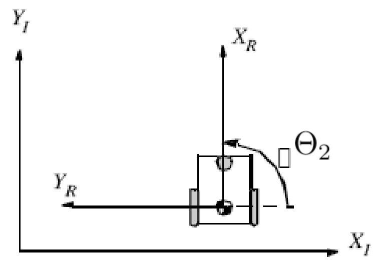
- The relation between the references frame is through the standard orthogonal rotation transformation:

$$R(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\theta = \theta_2 - \theta_1$$

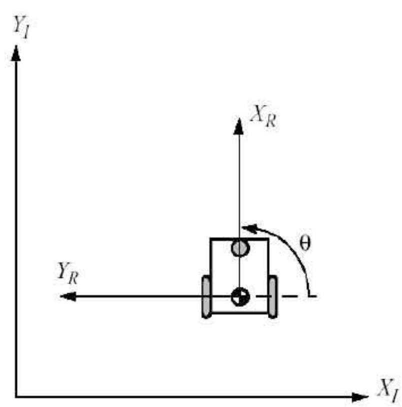
$$\dot{\xi}_R = R(\theta)\dot{\xi}_I$$





Example

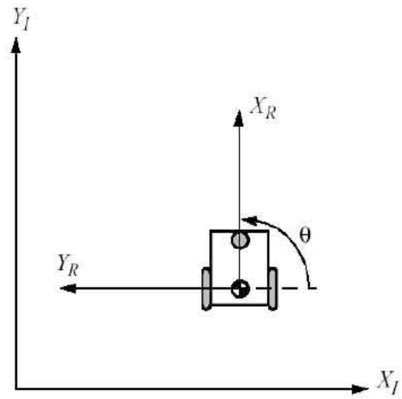
$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$\dot{\xi}_R = R(\frac{\pi}{2})$$

Example

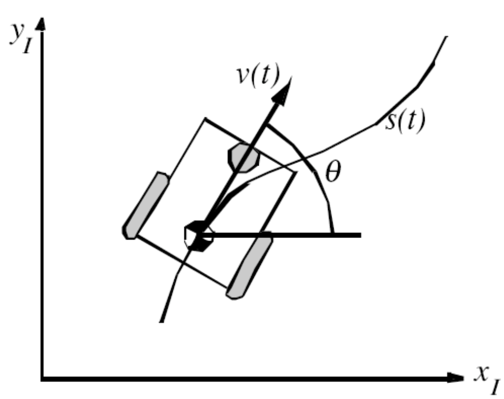
$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$\dot{\xi}_R = R(\frac{\pi}{2})\dot{\xi}_I = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{y} \\ -\dot{x} \\ \dot{\theta} \end{bmatrix}$$

- Goal:
  - Determine the robot speed $\dot{\xi} = \left[\begin{array}{ccc} \dot{x} & \dot{y} & \dot{\theta} \end{array} \right]^T$ as a function of wheel speed $\dot{\varphi}$, steering angle β, steering speed $\dot{\beta}$ and the geometric parameters of the robot.
  - Forward kinematics

$$\dot{\xi} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = f(\dot{\varphi}_1, \ldots, \dot{\varphi}_n, \beta_1, \ldots, \beta_m, \dot{\beta}_1, \ldots, \dot{\beta}_m)$$



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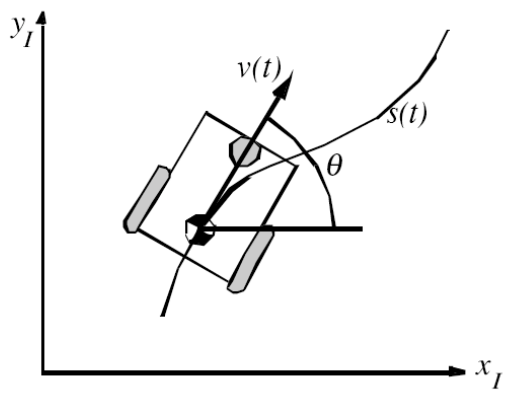
- Inverse kinematics

$$\begin{bmatrix} \dot{\varphi}_1 & \ldots & \dot{\varphi}_n & \beta_1 & \ldots & \beta_m & \dot{\beta}_1 & \ldots & \dot{\beta}_m \end{bmatrix}^T = f(\dot{x}, \dot{y}, \dot{\theta})$$



- Forward kinematics

$$\dot{\xi} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = f(\dot{\varphi}_1, \ldots, \dot{\varphi}_n, \beta_1, \ldots, \beta_m, \dot{\beta}_1, \ldots, \dot{\beta}_m)$$

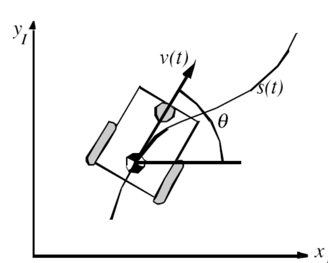
- Inverse kinematics

$$\begin{bmatrix} \dot{\varphi}_1 & \ldots & \dot{\varphi}_n & \beta_1 & \ldots & \beta_m & \dot{\beta}_1 & \ldots & \dot{\beta}_m \end{bmatrix}^T = f(\dot{x}, \dot{y}, \dot{\theta})$$

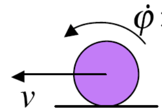
- Why not

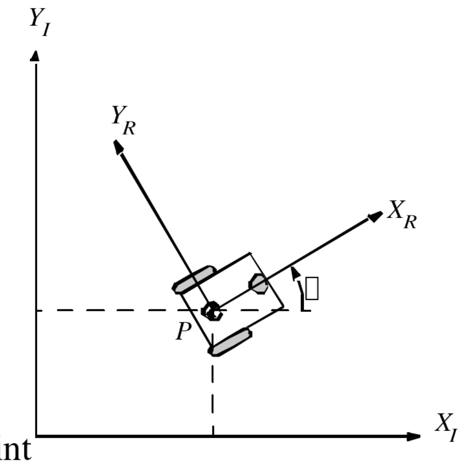
$$\begin{bmatrix} x \\ y \\ \theta \end{bmatrix} = f(\varphi_1, \ldots, \varphi_n, \beta_1, \ldots, \beta_m)$$

the relation is not straight forward. See later.



- Movement on a horizontal plane
- Point contact of the wheels
- Wheels not deformable
- Pure rolling
  - v = *0 at contact point*
- No slipping, skidding or sliding
- No friction for rotation around contact point
- Steering axes orthogonal to the surface
- Wheels connected by rigid frame (chassis)





Kinematic Models of wheel (rolling and sliding contraint)

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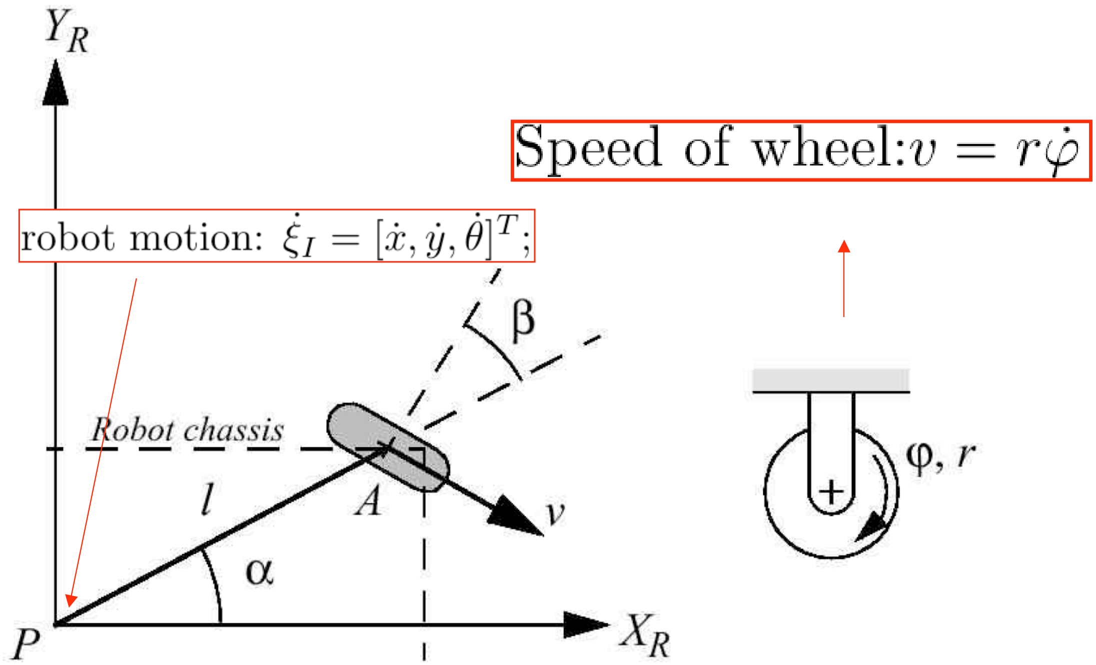
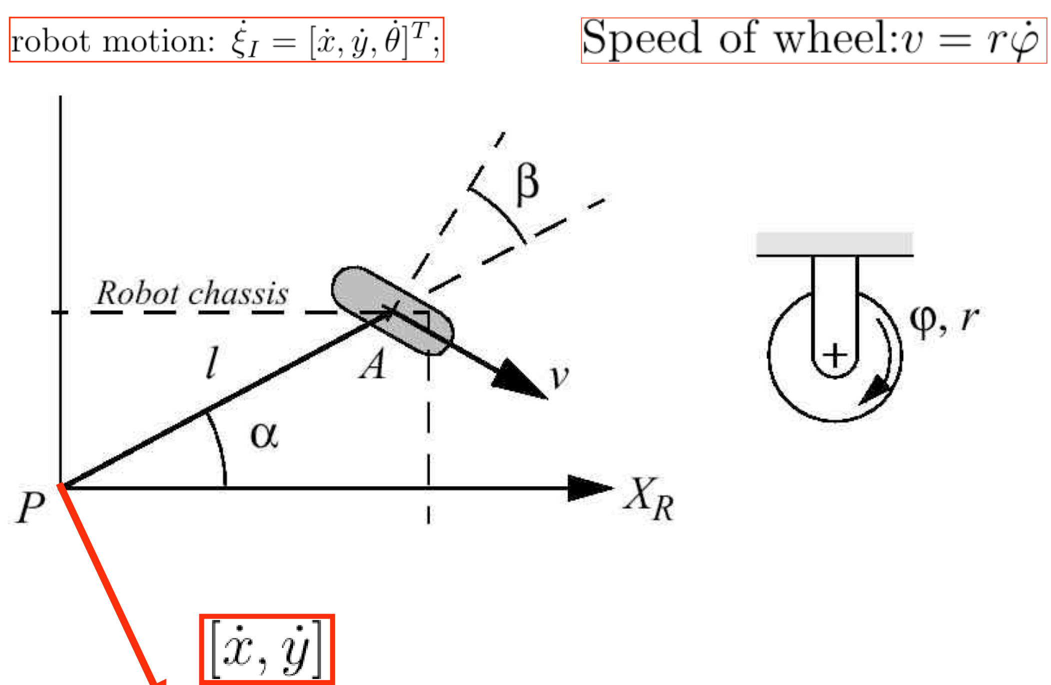
Mobil robot maneuverability



Speed of wheel:$v = r\dot{\varphi}$
robot motion: $\dot{\xi}_I = [\dot{x}, \dot{y}, \dot{\theta}]^T$;
Y_R
β
Robot chassis
φ, r
l
A
v
α
P
X_R



robot motion: $\dot{\xi}_I = [\dot{x}, \dot{y}, \dot{\theta}]^T$;
Speed of wheel:$v = r\dot{\varphi}$
β
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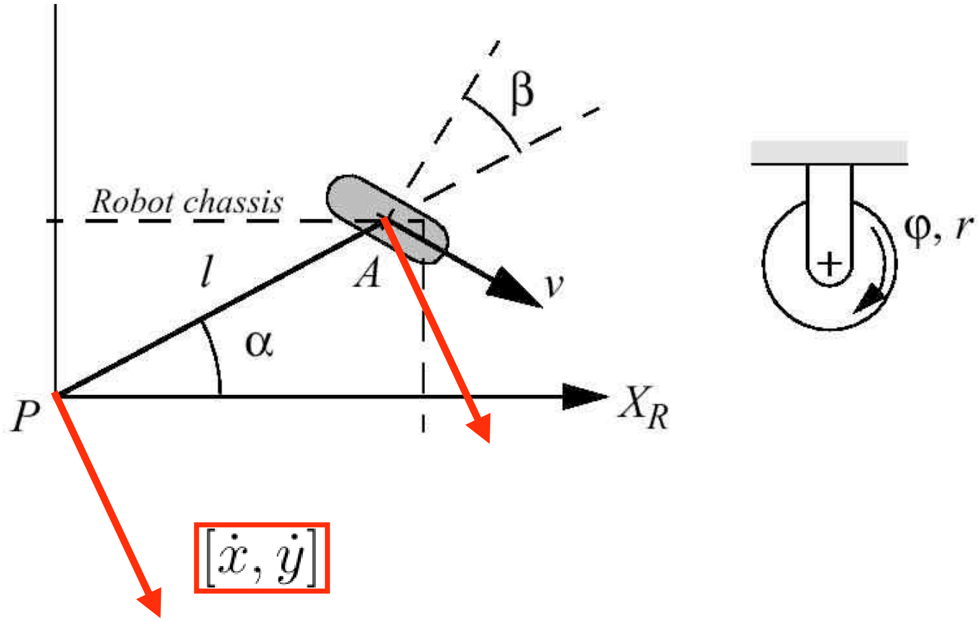
$[\dot{x}, \dot{y}]$

robot motion: $\dot{\xi}_I = [\dot{x}, \dot{y}, \dot{\theta}]^T$;

Speed of wheel: $v = r\dot{\varphi}$



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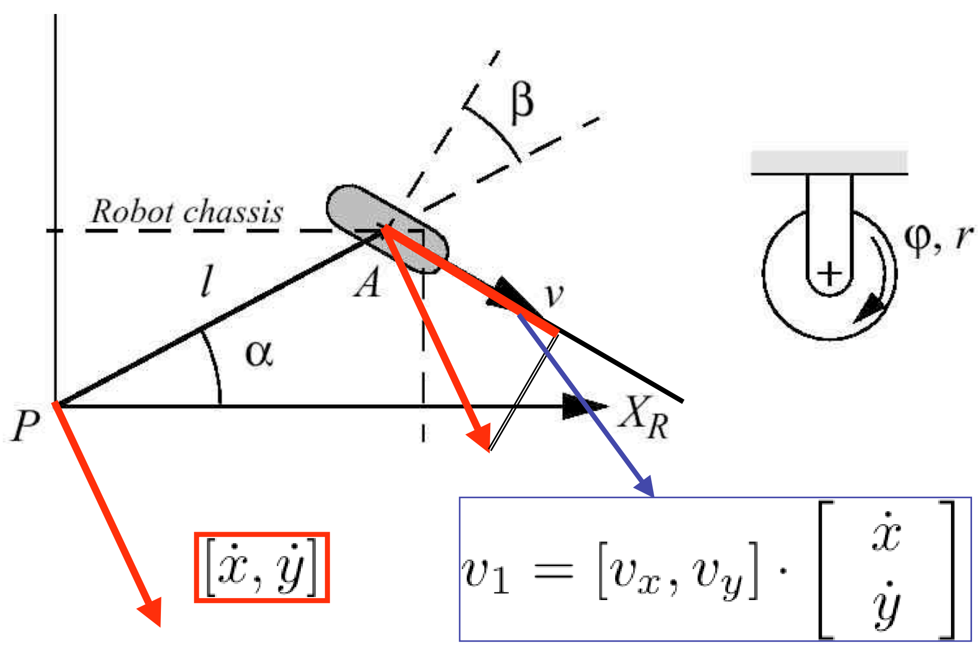

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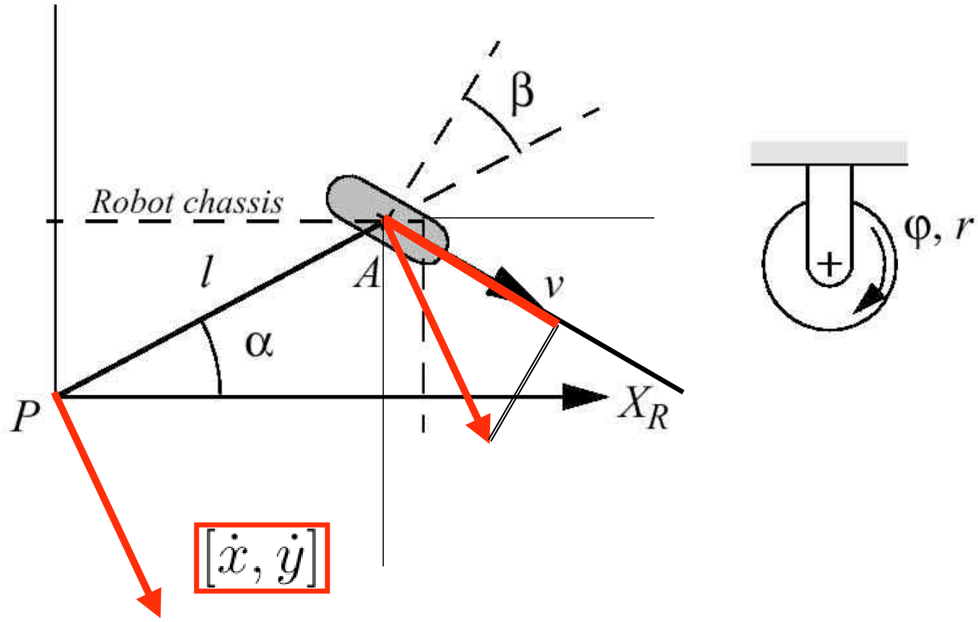

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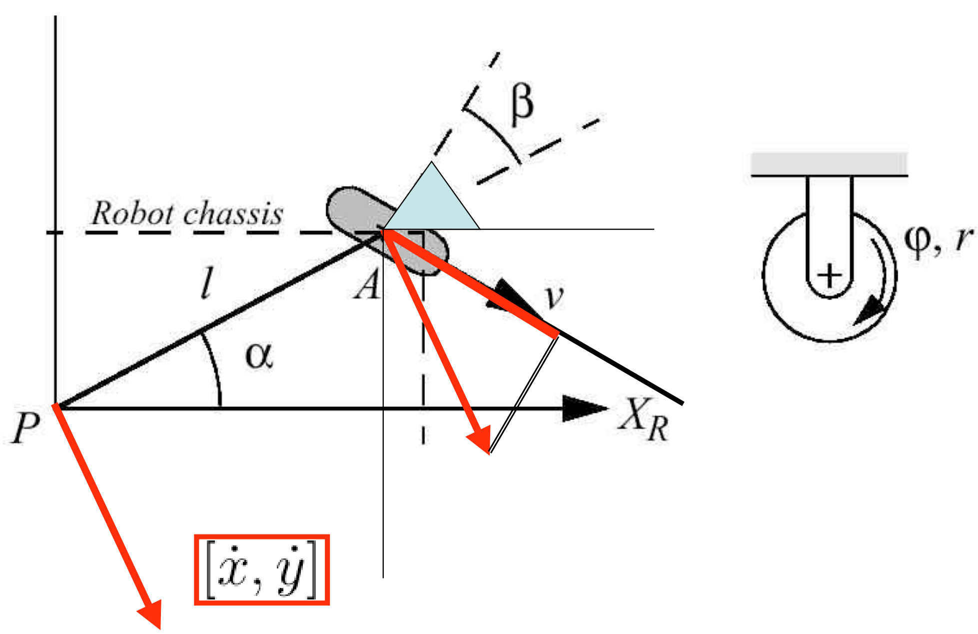

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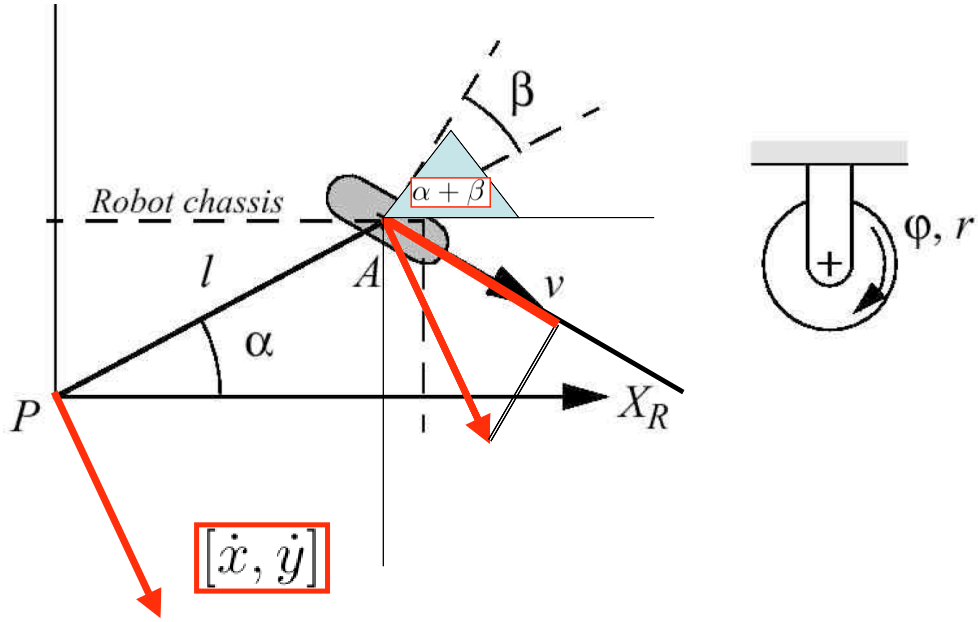

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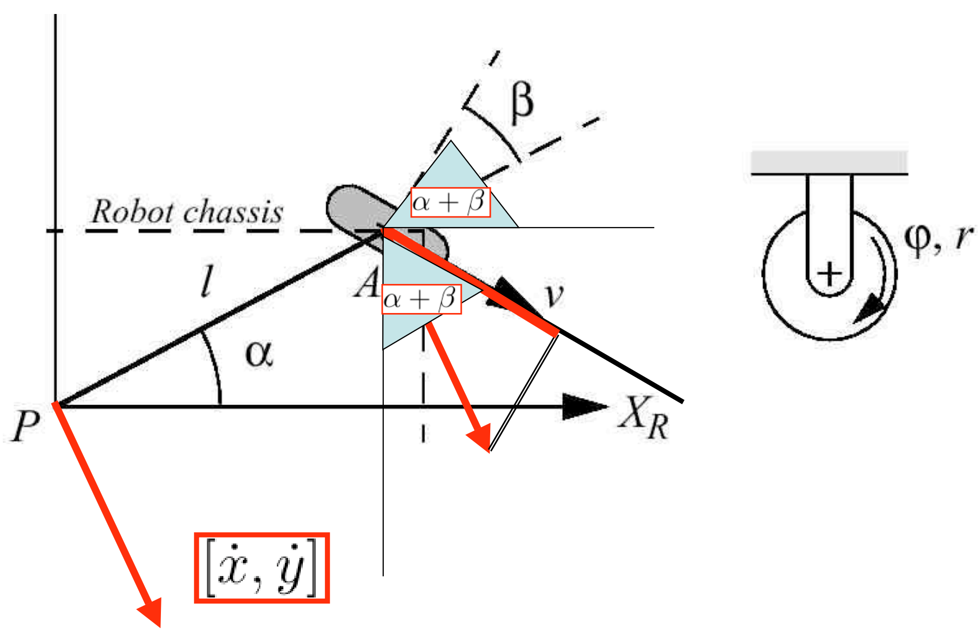

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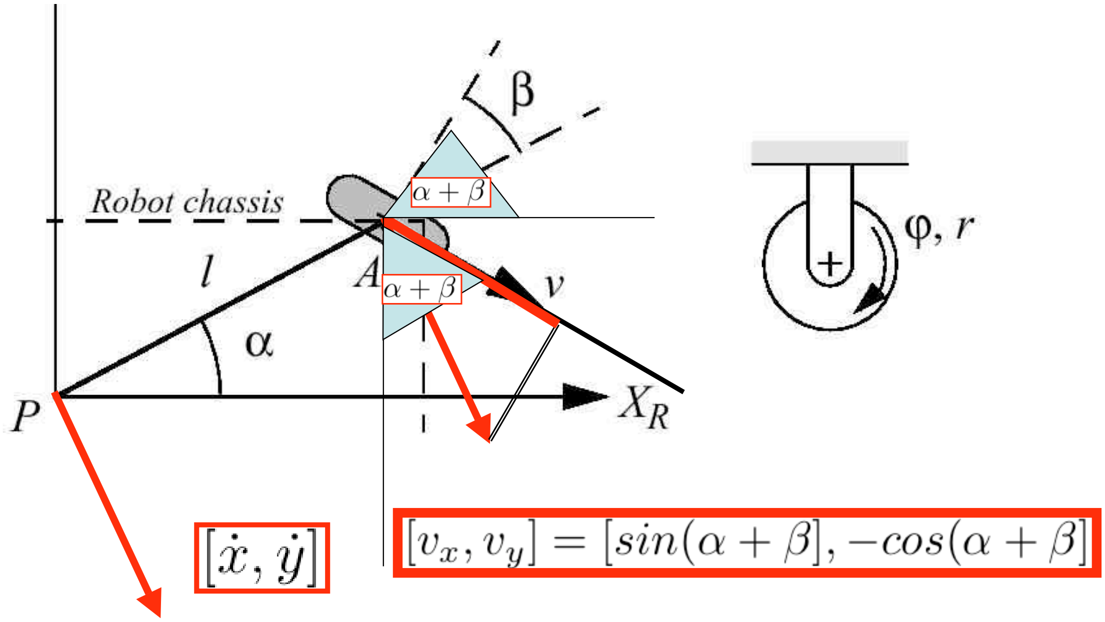

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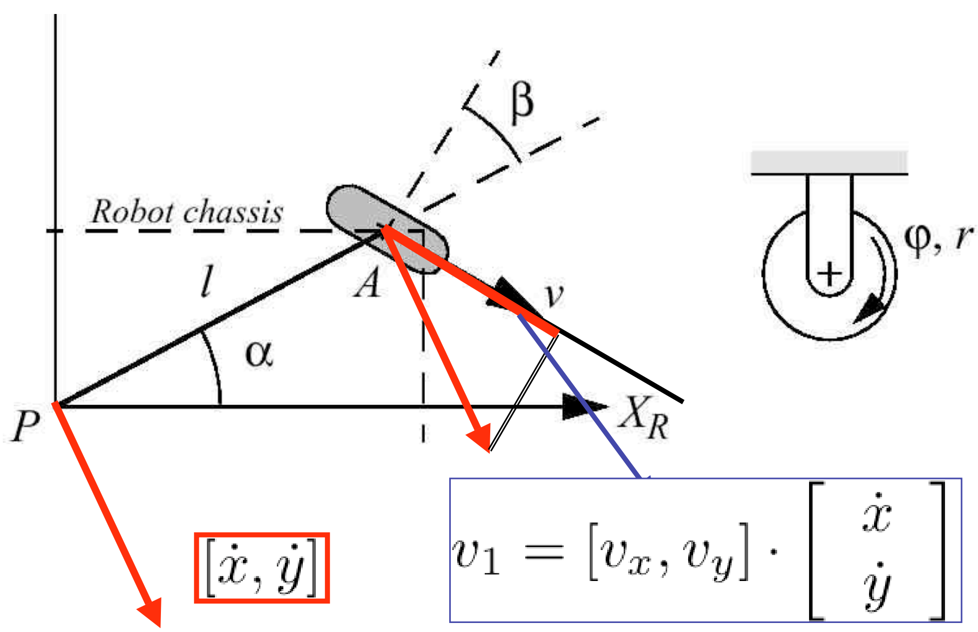

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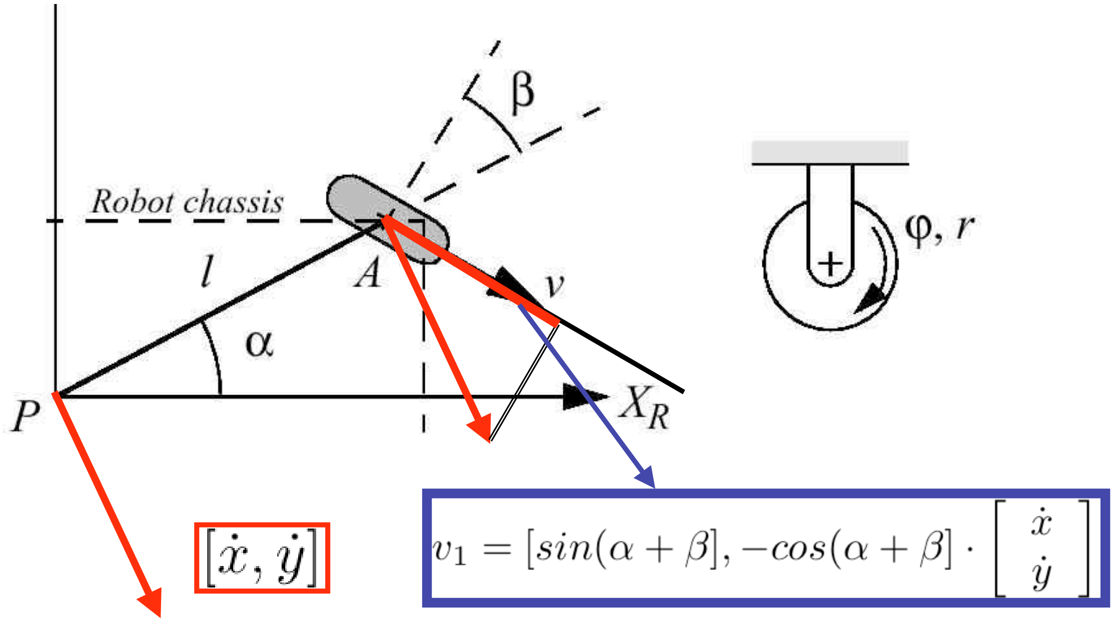

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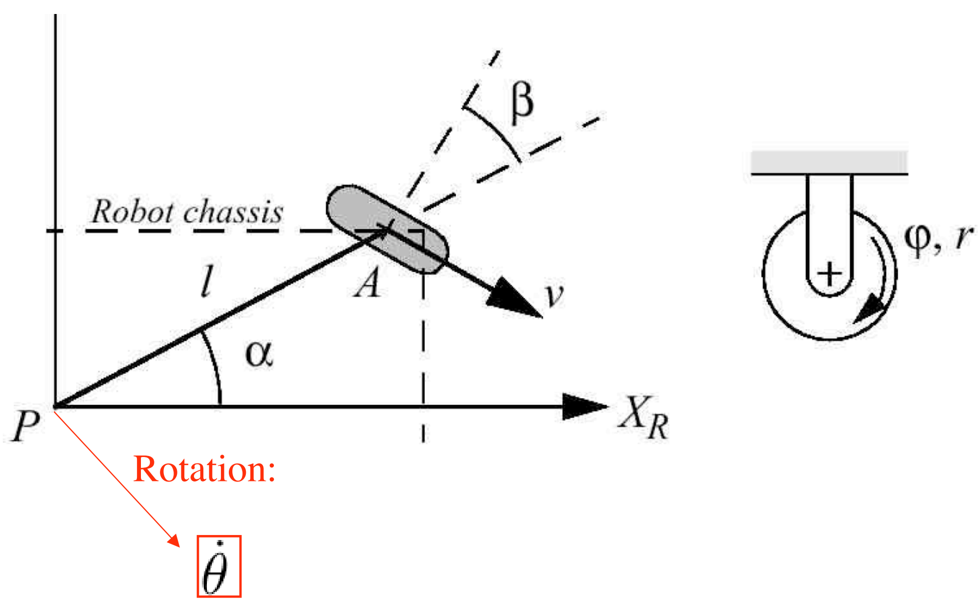

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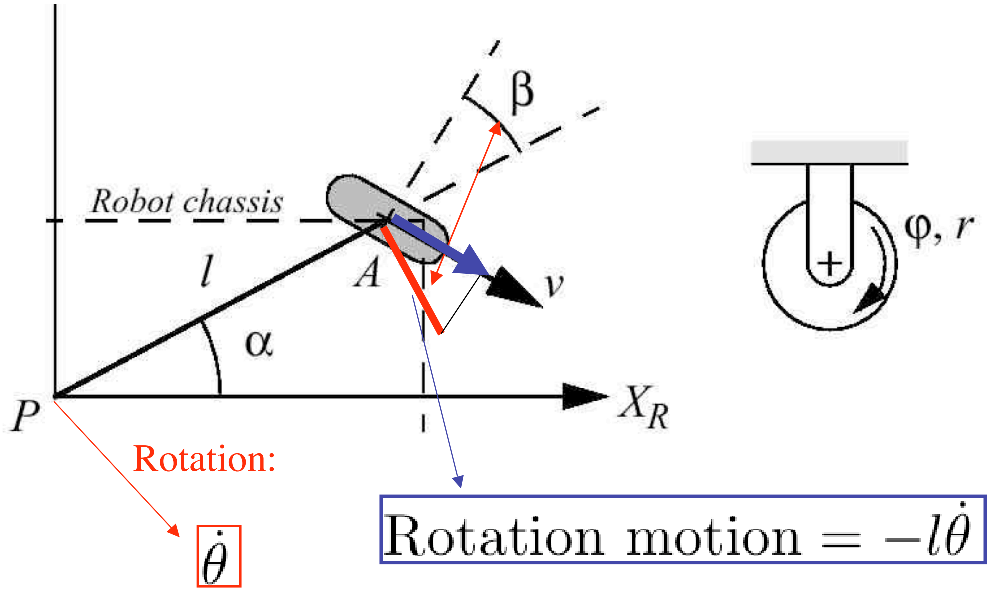

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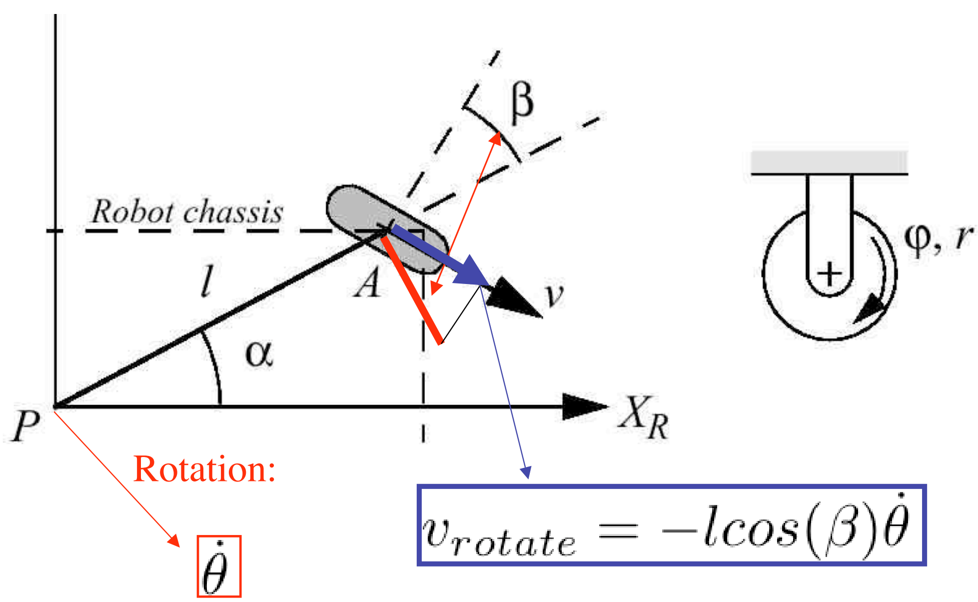


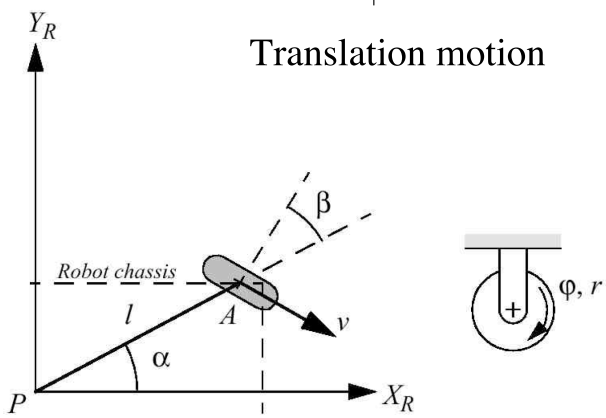
robot motion: $\dot{\xi}_I = [\dot{x}, \dot{y}, \dot{\theta}]^T$;

Speed of wheel: $v = r\dot{\varphi}$

$$v_1 = [sin(\alpha+\beta], -cos(\alpha+\beta)] \cdot \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + v_{rotate} = -lcos(\beta)\dot{\theta}$$

Translation motion

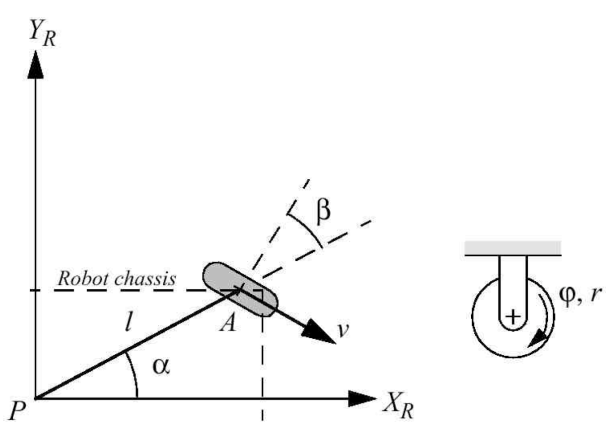
Rotation motion



robot motion: $\dot{\xi}_I = [\dot{x}, \dot{y}, \dot{\theta}]^T$;

Speed of wheel: $v = r\dot{\varphi}$

$$v = r\dot{\varphi} = [sin(\alpha+\beta]\dot{x}, -cos(\alpha+\beta)\dot{y}, -lcos(\beta)\dot{\theta}]$$



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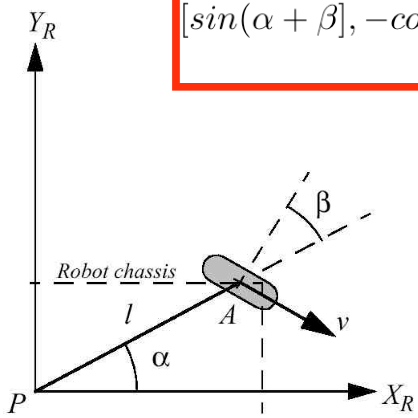
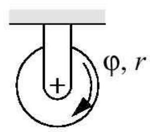
$$v = r\dot{\varphi} = [sin(\alpha+\beta]\dot{x}, -cos(\alpha+\beta)\dot{y}, -lcos(\beta)\dot{\theta}]$$

$$[sin(\alpha+\beta], -cos(\alpha+\beta), -lcos(\beta)] \cdot \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} - r\dot{\varphi} = 0$$





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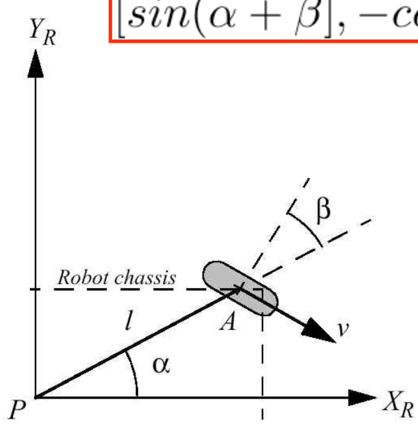
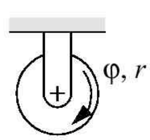


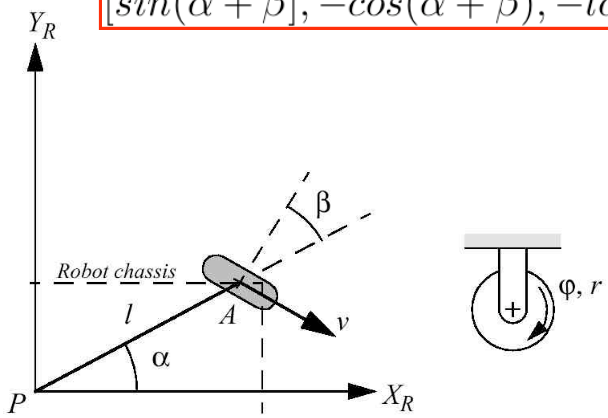


robot motion: $\dot{\xi}_I = [\dot{x}, \dot{y}, \dot{\theta}]^T$;

Speed of wheel: $v = r\dot{\varphi}$

$$v = r\dot{\varphi} = [sin(\alpha + \beta]\dot{x}, -cos(\alpha + \beta)\dot{y}, -lcos(\beta)\dot{\theta}]$$

$$[sin(\alpha + \beta], -cos(\alpha + \beta), -lcos(\beta)] \cdot R(\theta_R)\dot{\xi}_I - r\dot{\varphi} = 0$$



Transformation to the Reference frame

Motion caused by wheel motion, rolling constraints:

$$[sin(\alpha + \beta], -cos(\alpha + \beta), -lcos(\beta)] \cdot R(\theta_R)\dot{\xi}_I - r\dot{\varphi} = 0$$

Motion in the orthogonal plane must be 0, sliding constraints:

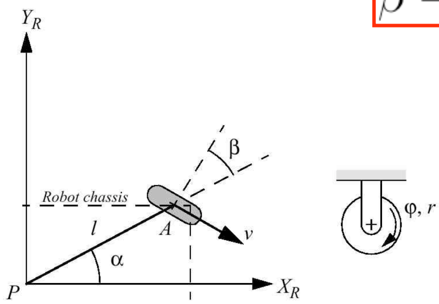
$$[cos(\alpha + \beta) \;\; sin(\alpha + \beta) \;\; l \cdot sin(\beta)]R(\theta_R)\dot{\xi}_I = 0$$

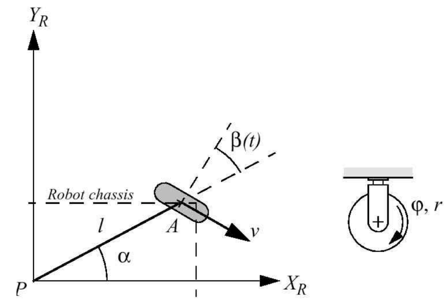
Fixed vs. Steered wheel:

$$[sin(\alpha+\beta], -cos(\alpha+\beta), -lcos(\beta)] \cdot R(\theta_R)\dot{\xi}_I - r\dot{\varphi} = 0$$

Same set contraints: but \beta is variable

$$\beta \rightarrow \beta(t)$$





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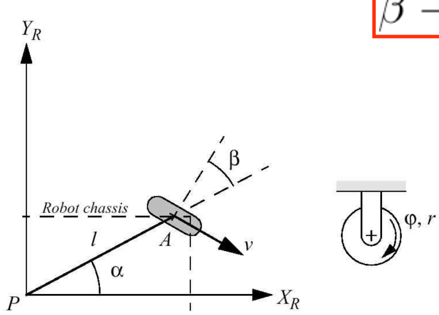
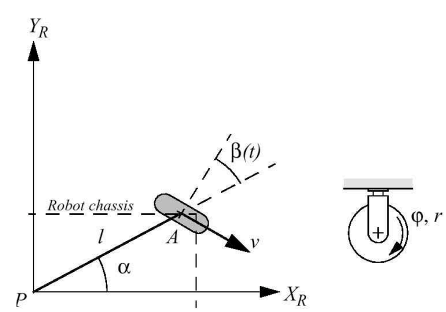
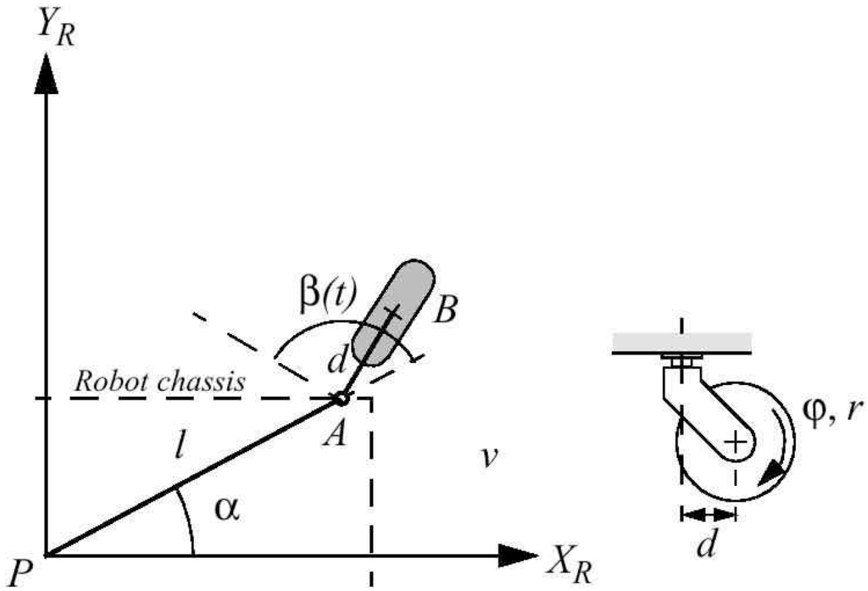
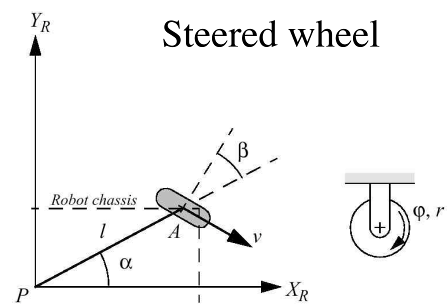




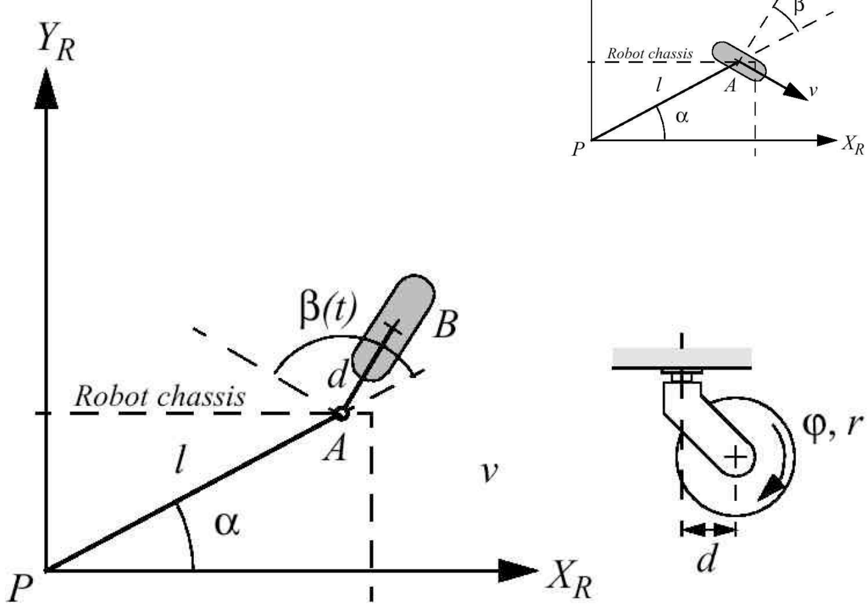
Caster wheel:



Steered wheel

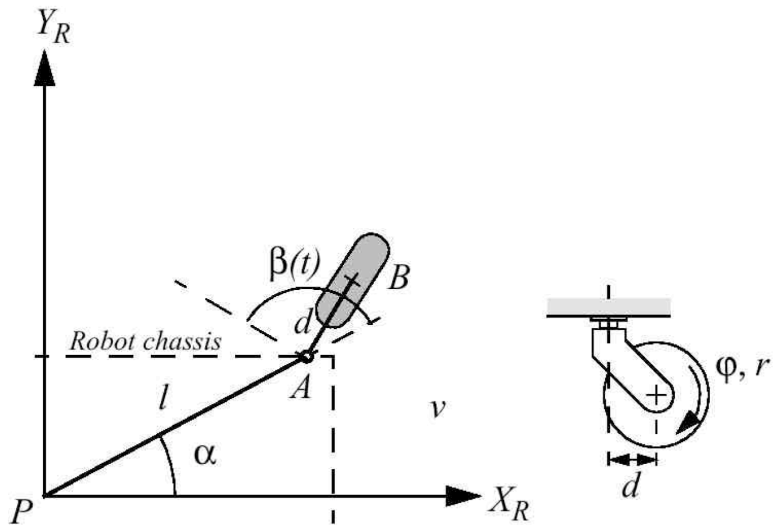


Caster wheel:



Caster wheel, rolling constraint(same as steered case):

$$[sin(\alpha + \beta], -cos(\alpha + \beta), -lcos(\beta)] \cdot R(\theta_R)\dot{\xi}_I - r\dot{\varphi} = 0$$

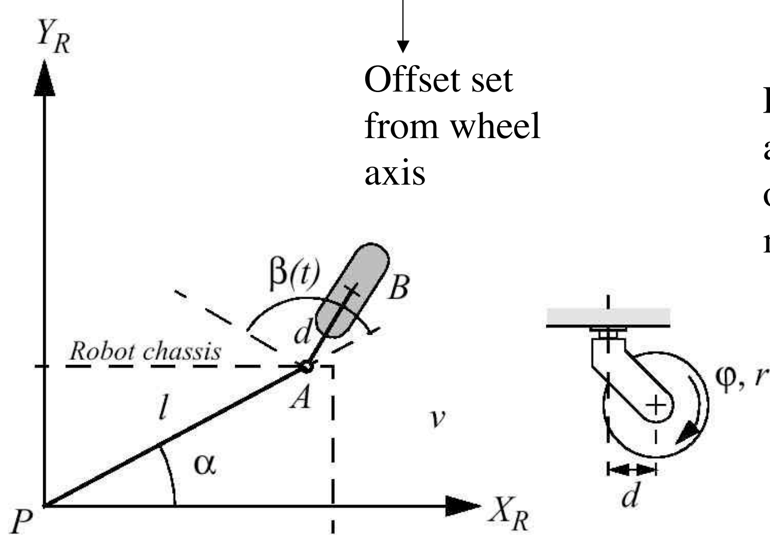


… But the sliding constraint(is different):

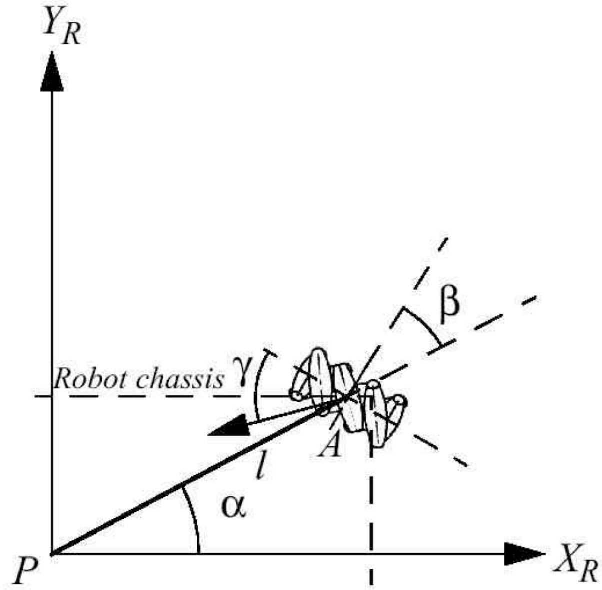
$$[cos(\alpha + \beta)\ sin(\alpha + \beta]\ d + l \cdot sin(\beta)]R(\theta_R)\dot{\xi}_I + d\dot{\beta} = 0$$

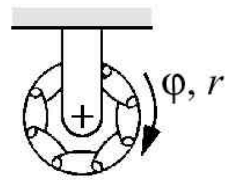
Offset set from wheel axis

Rotation can be arbitrary: allow omidirectional motion



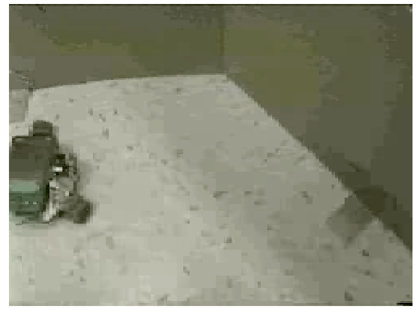
Swedish Wheel





Swedish 90

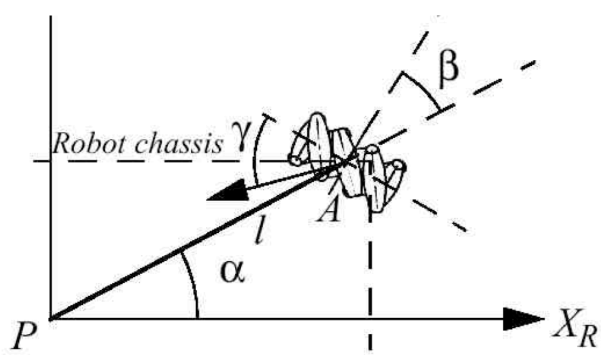
Palm Pilot Robot, CMU



Palm Pilot Robot, CMU

Swedish wheel, rolling constraint:

$$[sin(\alpha + \beta + \gamma), \quad -cos(\alpha + \beta + \gamma), \quad -lcos(\beta + \gamma)]R(\theta_R)\dot{\xi}_I - r\dot{\varphi}cos(\gamma) = 0$$

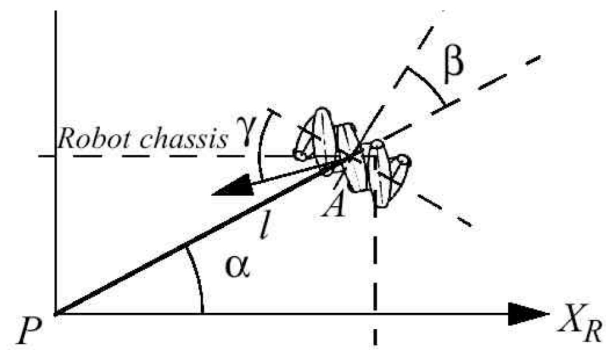


Swedish wheel, rolling constraint:

$$[sin(\alpha+\beta\boxed{+\gamma}], \quad -cos(\alpha+\beta\boxed{+\gamma}), \quad -lcos(\beta\boxed{+\gamma})]R(\theta_R)\dot{\xi}_I - r\dot{\varphi}\boxed{cos(\gamma)} = 0$$

Sliding constraints:

$$[cos(\alpha+\beta\boxed{+\gamma}) \; sin(\alpha+\beta\boxed{+\gamma}] \; l\cdot sin(\beta\boxed{+\gamma})]R(\theta_R)\dot{\xi}_I - \boxed{r\dot{\varphi}sin(\gamma)} - \boxed{r_{sw}\dot{\varphi_{sw}}} = 0$$



Kinematic Models of wheel (rolling and sliding contraints)

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Mobil robot maneuverability

Suppose we have a total of $N=N_f+N_s$ standard wheels

- *We can develop the equations for the constraints in matrix forms :*
- *Rolling*

$$J_1(\beta_s)R(\theta)\dot{\xi}_I + J_2\dot{\varphi} = 0 \qquad \varphi(t) = \begin{bmatrix} \varphi_f(t) \\ \varphi_s(t) \end{bmatrix}$$

$$J_1(\beta_s) = \underbrace{\begin{bmatrix} J_{1f} \\ J_{1s}(\beta_s) \end{bmatrix}}_{(N_f+N_s)\times 3} = diag(r_1 \cdots r_N)$$

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$$J_1(\beta_s)R(\theta)\dot{\xi}_I + J_2\dot{\varphi} = 0 \qquad \varphi(t) = \begin{bmatrix} \varphi_f(t) \\ \varphi_s(t) \end{bmatrix}$$

- *Lateral movement*

$$C_1(\beta_s)R(\theta)\dot{\xi}_I = 0 \qquad C_1(\beta_s) = \underbrace{\begin{bmatrix} C_{1f} \\ C_{1s}(\beta_s) \end{bmatrix}}_{(N_f+N_s)\times 3}$$

Suppose we have a total of $N = N_f + N_s$ standard wheels

- *We can develop the equations for the constraints in matrix forms :*
- *Rolling*

$$J_1(\beta_s)R(\theta)\dot{\xi}_I + J_2\dot{\varphi} = 0 \qquad \varphi(t) = \begin{bmatrix} \varphi_f(t) \\ \varphi_s(t) \end{bmatrix}$$

- *Lateral movement*

$$C_1(\beta_s)R(\theta)\dot{\xi}_I = 0$$

Examples: differential drive + omidirectional drive

Mobile Robot Maneuverability: Degree of Mobility

- To avoid any lateral slip the motion vector $R(\theta)\dot{\xi}_I$ has to satisfy the following constraints:

$$C_{1f}R(\theta)\dot{\xi}_I = 0$$

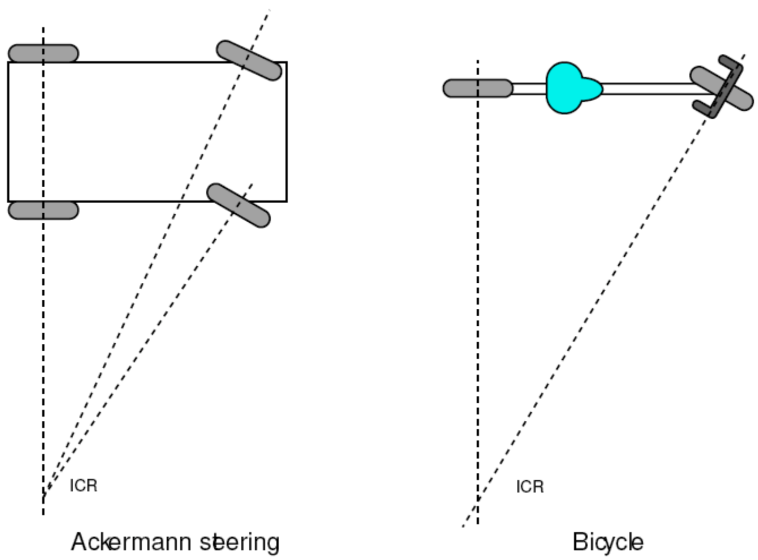
$$C_{1s}(\beta_s)R(\theta)\dot{\xi}_I = 0$$

$$C_1(\beta_s) = \begin{bmatrix} C_{1f} \\ C_{1s}(\beta_s) \end{bmatrix}$$

- Mathematically:
  - $R(\theta)\dot{\xi}_I$ *must belong to the* null space *of the projection matrix* $C_1(\beta_s)$
  - Null space *of* $C_1(\beta_s)$ *is the space* N *such that for any vector* n *in* N

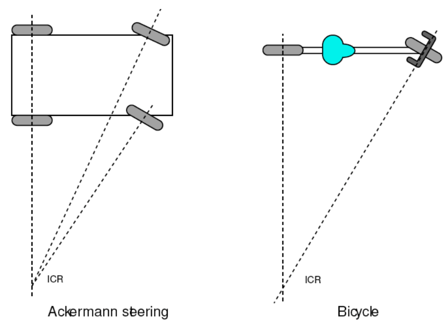
$$C_1(\beta_s) \times n = 0$$

  - *Geometrically this can be shown by the* Instantaneous Center of Rotation (ICR)



Mobile Robot Maneuverability: More on Degree of Mobility

- Robot chassis kinematics is a function of the set of *independent constraints* $rank[C_1(\beta_s)]$
  - *the greater the rank of* , $C_1(\beta_s)$ *the more constrained is the mobility*



The $rank(C_1)$ defines the number of independent constraints

- *no standard wheels* $rank[C_1(\beta_s)] = 0$
- *all direction constrained* $rank[C_1(\beta_s)] = 3$

The degree of mobility is defined by the dimensionality of the null space of C_1 which for a mobile platform is equal to:

$$\delta_m = dim(null(C_1)) = 3 - rank(C_1)$$

- *no standard wheels* $rank[C_1(\beta_s)] = 0$
- *all direction constrained* $rank[C_1(\beta_s)] = 3$

The degree of mobility is defined by the dimensionality of the null space of C_1 which for a mobile platform is equal to:

$$\delta_m = dim(null(C_1)) = 3 - rank(C_1)$$

Examples:

Robot	δ_m
Differential drive	2
Bicycle	1

- Steerability is the number of independent DOF that can be controlled

$$\delta_s = rank(C_{1s})$$

- Similarly the degree of maneuverability is defined as

$$\delta_M = \delta_m + \delta_s$$

- Degree of Maneuverability

$$\delta_M = \delta_m + \delta_s$$

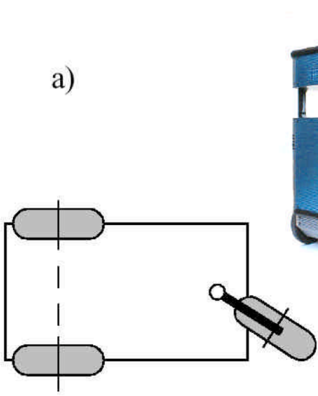
   - *Two robots with same δ_M are not necessary equal*
   - *Example: Differential drive and Tricycle (next slide)*

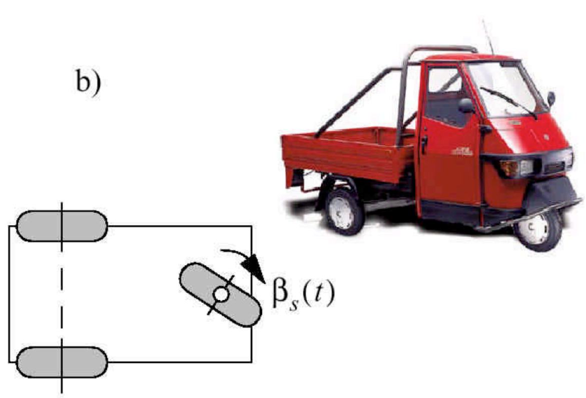
   - *For any robot with $\delta_M = 2$ the ICR is always constrained to* lie on a line
   - *For any robot with $\delta_M = 3$ the ICR is not constrained an can* be set to any point on the plane

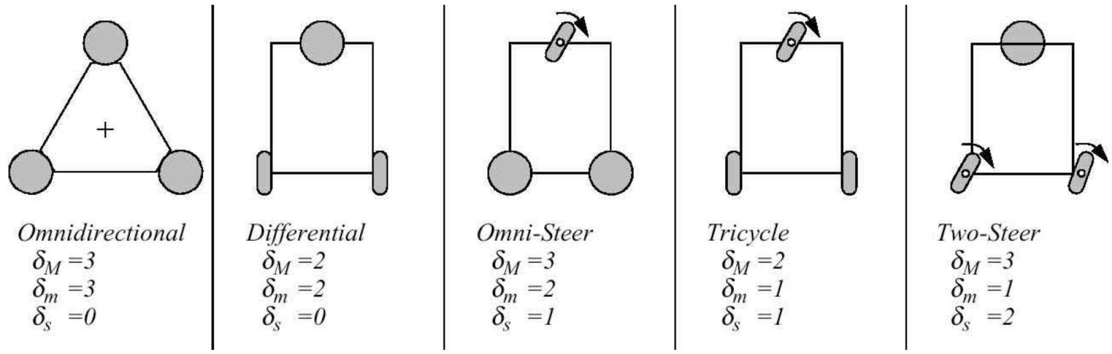
- The Synchro Drive example: $\delta_M = \delta_m + \delta_s = 1 + 1 = 2$

Mobile Robot Maneuverability: Wheel Configurations

- Differential Drive Tricycle

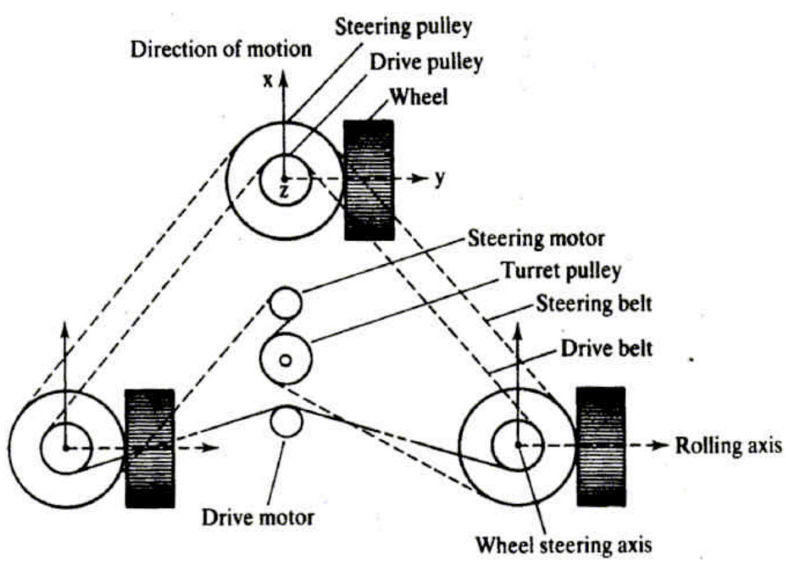


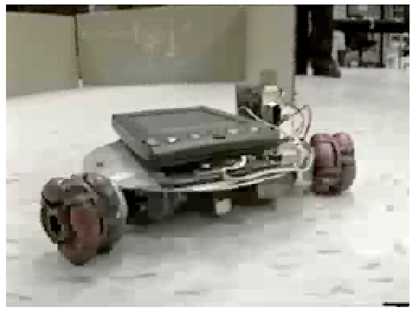




Synchro Drive

$$\delta_M = \delta_m + \delta_s = 1 + 1 = 2$$





Palm Pilot Robot, CMU